Exponential Differentiation

ID: 8980

Time required
45 minutes

Activity Overview

Students will start this activity by observing an interesting property related to the graph of a tangent to $y = e^x$. This property will allow them to readily see that the derivative of $y = e^x$ is itself. This result will be confirmed using limits. Students will then determine that $\lim_{h \to 0} \frac{e^h - 1}{h}$ equals 1 and use this fact to evaluate this derivative analytically. Finally, students will use the chain rule to find the derivative of exponential functions of any base.

Topic: Formal Differentiation

- *Derive the Exponential Rule and the Generalized Exponential Rule for differentiating exponential functions.*
- *Use the Limit command to show $\lim_{h \to 0} \frac{e^{a+h} - e^a}{h} = e^a$ and verify the Exponential Rule for differentiation.*
- *Graph the function $f(x) = e^x$ and measure the slope to the graph at any point $x = a$ to verify that $f'(x) = f(a)$.*

Teacher Preparation and Notes

- *This investigation offers an opportunity for students to find that the derivative of $y = e^x$ is itself. This is the only function for which this is true and the graphical introduction to this activity allows students to visualize this unique property.*
- *Students should know how to write an expression with limits that gives the derivative of a function at a specific value. They should also know how to set up a similar expression that will give the derivative as a function of x. Finally, students will need to use the chain rule to find the derivative of exponential functions with any base.*
- *Students will manipulate pre-made sketches, rather than create their own constructions. Therefore, a basic working knowledge of the TI-Nspire handheld is sufficient.*
- *Notes for using the TI-Nspire™ Navigator™ System are included throughout the activity. The use of the Navigator System is not necessary for completion of this activity.*
- ***To download the student and solution TI-Nspire documents (.tns files) and student worksheet, go to education.ti.com/exchange and enter for "8980" in the keyword search box.***

Associated Materials

- *ExponentialDifferentiation_Student.doc*
- *ExponentialDifferentiation.tns*
- *ExponentialDifferentiation_Soln.tns*

Suggested Related Activities

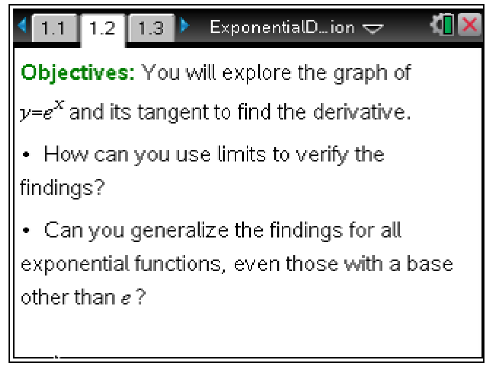
To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword search box.

- *Exponentially Fast Derivative (TI-Nspire CAS technology) — 11612*

One focus question defines the objective of this activity:

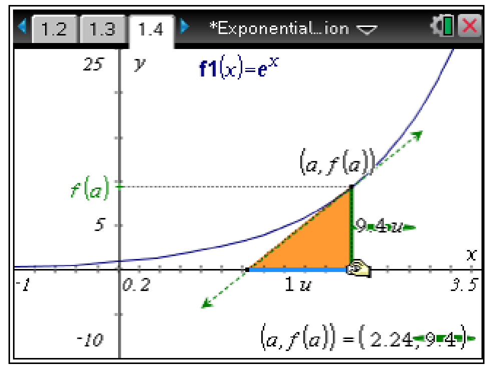
How can you use the tangent line to $y = e^x$ to find the derivative of $y = e^x$?

Explain to students that the questions posed on page 1.2 will be explored during this activity and are central to the concept of exponential differentiation.



Problem 1 – Investigating the Graph of $y = e^x$

Students begin working cooperatively on page 1.4, with the graph of the function **f1**(x) = e^x and a moveable tangent to **f1**(x).The questions posed on the student worksheet are intended to lead them to the conclusion that the derivative of **f1**(x) = e^x is itself.



Student Worksheet Solutions

1. The triangle is constructed such that its vertices are (i) the point of tangency, (ii) the intersection of the tangent and the x-axis, and (iii) the intersection of the x-axis and the perpendicular line drawn from the point of tangency to the x-axis.
2. By finding the ratio of the vertical leg of the right triangle to the horizontal leg of the right triangle.
3. The vertical leg (rise) of the right triangle changes and is equal to **f1**(a) = e^a. The horizontal leg (run) is always equal to 1.

NAVIGATOR OPPORTUNITY #1: Live Presenter –

TI-Nspire Navigator Opportunity: *Live Presenter*   **See Note 1 at the end of this lesson.**

4. It appears that the slope of the triangle is equal to **f1**(a) = e^a which is the distance from the point of tangency to the x-axis.

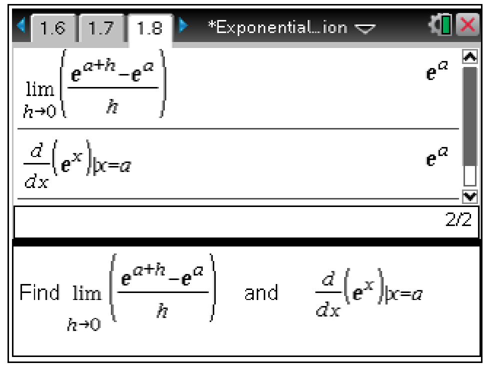
TI-Nspire Navigator Opportunity: *Quick Poll*   **See Note 2 at the end of this lesson.**

5. The slope of the tangent at $x = a$ is equal to **f1**(a) = e^a.

6. The derivative of $y = e^x$ is itself, or $\frac{d}{dx}\left(e^x\right) = e^x$.

7. By evaluating a limit, the result from the previous section can be verified. The calculator screen at right shows that, by definition, the derivative of **f1**(x) = e^x at $x = a$ is itself.

8. The second calculation shown at right demonstrates that the derivative of **f1**(x) = e^x is **f1'**(x) = e^x.



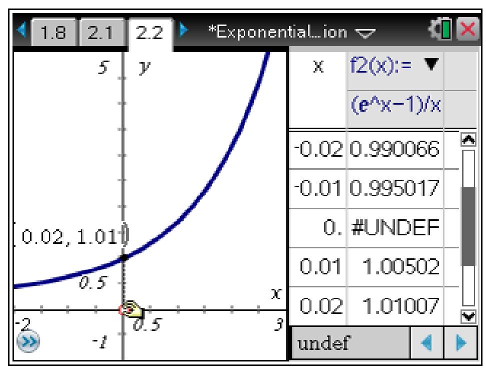
Problem 2 – Generalizing Your Findings

The limit expression $\frac{d}{dx}\left(e^x\right) = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$ can be simplified using a variety of properties from algebra, the right side of this equation can be simplified as follows:

$$\begin{aligned}\frac{d}{dx}\left(e^x\right) &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h} \quad \text{because } a^m\cdot a^n = a^{m+n}\\ &= e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h}.\end{aligned}$$

9. Because $\frac{d}{dx}\left(e^x\right) = e^x$, the limit $\lim_{h\to 0}\frac{e^h-1}{h}$ must be equal to 1.

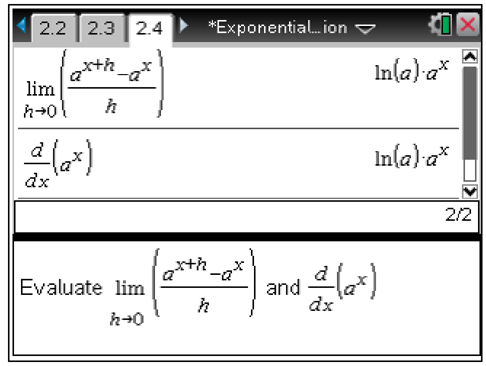
10. Students will investigate this limit graphically and numerically on page 2.2 as shown on the screen at right. Notice that the left and right limits (from the table appear to be approaching 1, even though the expression itself is undefined at $x = 0$.



TI-Nspire Navigator Opportunity: *Screen Capture*

See Note 3 at the end of this lesson.

11. Students now extend their investigation to include any exponential function, $y = a^x$ where $a > 0$ and $a \neq 0$. The screen at right shows that $\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x$.



12. Using the chain rule, students can prove that $\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x$ as follows:

$$\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(e^{x\ln a}\right) \qquad \text{because } e^{x\ln a} = e^{\ln a^x} = a^x.$$

$$= e^{x\ln a}\frac{d}{dx}\left(x\ln a\right) \qquad \text{the chain rule}$$

$$= e^{x\ln a} \bullet \ln a$$

$$= a^x \ln a \qquad \text{because } e^{x\ln a} = a^x.$$

This result is confirmed by using the **derivative** command as shown on the screen at right.

TI-Nspire Navigator Opportunities

Note 1
Problem 1, *Live Presenter*
After students discuss their answer to Question 3, have a volunteer explain their answer. You can listen as the partners discuss their answer to assign a student with a good explanation to demonstrate with Live Presenter.

Note 2
Problem 1, *Quick Poll*
Send page1.7 to students and have students answer the question.

Note 3
Problem 3, *Screen Capture*
Use Screen Capture to discuss and compare students screenshots of page 2.2. What is this the graph of? (Answer: The derivative of e^x.)