Riemann Sums

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Abstract: This activity is an introduction to integration. Students use their calculator to investigate Riemann sums and learn how to construct them. Then they take the limit of two Riemann sums and find out that the limits exist and are both the same. They compare this limit to the definite integral of the function.

NCTM Principles and Standards:
Algebra standards
a) analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
b) use symbolic algebra to represent and explain mathematical relationships;
c) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
d) draw reasonable conclusions about a situation being modeled.

Geometry standards: Analyze characteristics and properties of two- and three-dimensional geometric shapes and mathematical about geometric relationships
Measurement standards: understand and use formulas for the area, surface area, and volume of geometric figures, including cones, spheres, and cylinders;
Problem Solving Standard: build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.
Reasoning and Proof Standard
a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

Key topic: Applications of Integrals- Riemann sums

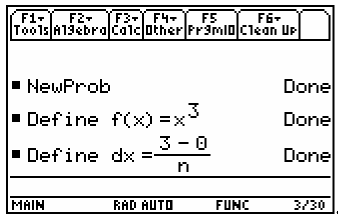
Degree of Difficulty: Elementary
Needed Materials: TI-89 calculator

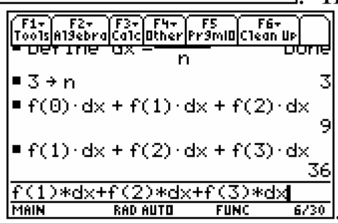
Situation: The definite integral of a continuous function is defined as the limit of any Riemann sum of that function as the number of intervals gets arbitrarily large. What is a Riemann sum? Suppose we have a continuous function f(x) with a domain [a, b]. Divide the interval into n equal width subintervals. If we let Δx stand for that width, then $\Delta x = \frac{b-a}{n}$. Let x_0 be the left endpoint of the first subinterval (so x_0 = a), x_1 be the left endpoint of the second subinterval (which means that it is also right endpoint of the first subinterval , etc. and let x_n be the right endpoint of the n^{th} subinterval (so x_n=b). Then we can construct two particular Riemann sums for the function:

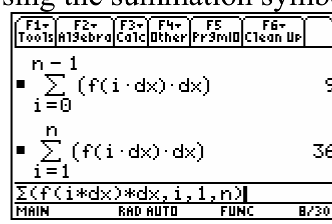
The Left sum = $f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$

The Right sum = $f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$

Each of these terms represents the area of a rectangle. We can use our calculator to compute these.

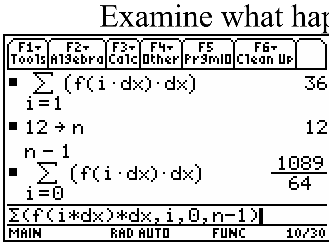
Define $f(x) = x^3$ and consider the interval [0, 3], then $\Delta x = \frac{3-0}{n}$. We'll use the variable dx to represent Δx: 

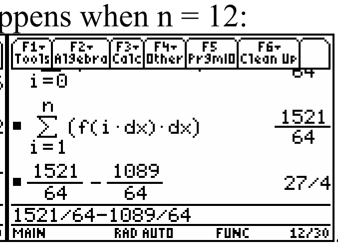
. If n = 3, we can easily compute the left sum and the right sum: 

. Using the summation symbol from the calculus menu makes expressing these sums easier: 

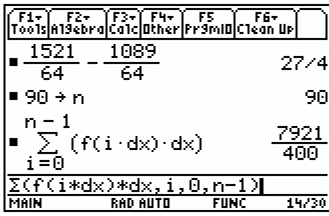
. Let's examine the syntax of these commands carefully. Consider the left sum: $\sum_{i=0}^{n-1}(f(i \cdot dx) \cdot dx)$. Here we are summing n terms, although in our example n = 3. So dx = 1. The first term evaluates the expression at i=0, the second at i = 1dx = 1, and the third term (when i = 3-1=2) x has a value of 2*dx. These are the same terms that we evaluated explicitly above. The summation notation makes it easier to express Riemann sums.

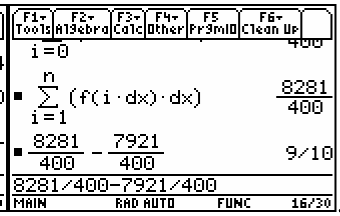
Examine what happens when n = 12:





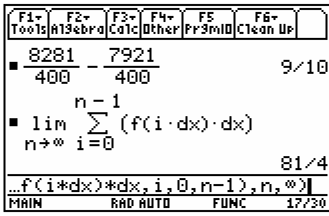
. Now the left and right sums are fairly close to each other. What if we picked a larger value for n, say 90?

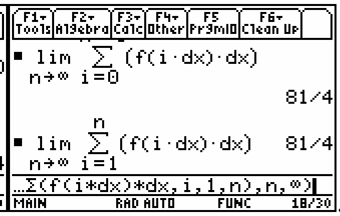




. These values are quite close, now.

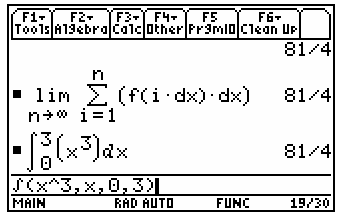
Let's now consider the limit as n $\rightarrow\infty$ of these two sums:

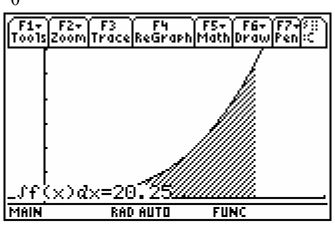




. They are both the same!

According to the definition of a definite integral, each of these limits shows that

$\int_0^3 x^3 dx = \frac{81}{4}$. Let's check that with our calculator: 



Note: If we had wanted to use a different interval, say [5, 13], we would have defined $dx = \frac{13-5}{n}$ and have the following in our summation statement: $f(5 + i \cdot dx) \cdot dx$. As a further example, if we had wanted to use the interval [π, 7] we would have defined $dx = \frac{7-\pi}{n}$ and have the following in our summation statement: $f(\pi + i \cdot dx) \cdot dx$.