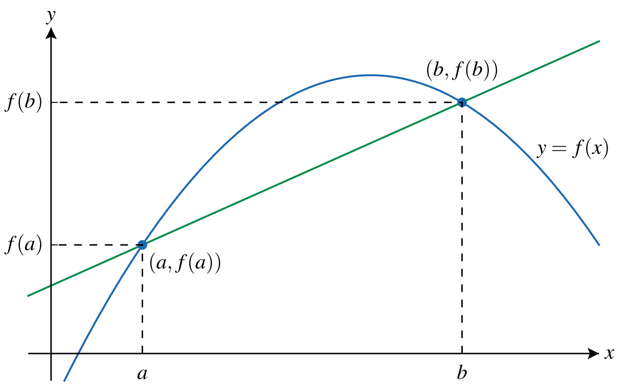
Thursday Night PreCalculus, November 30, 2023

Difference Quotients and Average Rates of Change

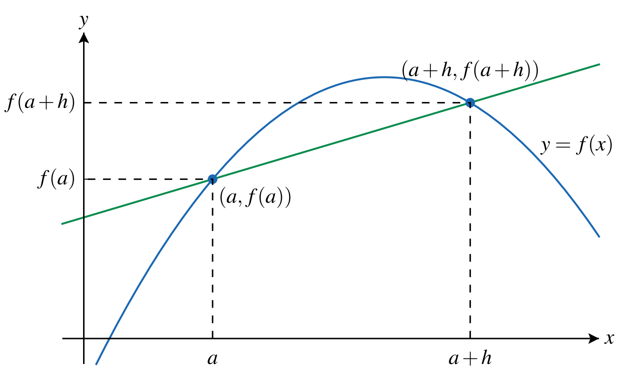
A Few Geometric Interpretations

Average rate of change: the change in y divided by the change in x.

$$\text{Ave Rate of Change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}$$



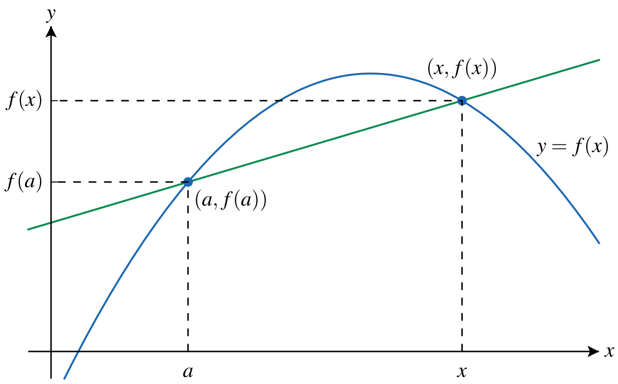
Difference Quotient: A measure of the average rate of change of a function over an interval of length h.



$$\text{DQ} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

$$\text{DQ} = \frac{f(x+h) - f(x)}{h}$$

Another perspective:



$$\text{DQ} = \frac{f(x) - f(a)}{x - a}$$

Problems

1. For each of the following functions, simplify the expression

$$\frac{f(x+h)-f(x)}{h}, \qquad h \neq 0$$

as far as possible.

(a) $f(x) = 3x^2 - 5x$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{3(x+h)^2 - 5(x+h) - (3x^2 - 5x)}{h} \\ &= \frac{3(x^2 + 2xh + h^2) - 5x - 5h - 3x^2 + 5x}{h} \\ &= \frac{3x^2 + 6xh + 3h^2 - 5x - 5h - 3x^2 + 5x}{h} \\ &= \frac{6xh + 3h^2 - 5h}{h} = \frac{h(6x + 3h - 5)}{h} \\ &= 6x + 3h - 5\end{aligned}$$

Note: What is the average rate of change for consecutive equal-length intervals?

Interval	Difference Quotient
$[a, a+h]$	$6a + 3h - 5$
$[a+h, a+2h]$	$6a + 9h - 5$
$[a+2h, a+3h]$	$6a + 15h - 5$

(b) $f(x) = \sqrt{x^2 - 1}$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{(x+h)^2 - 1} - \sqrt{x^2 - 1}}{h} \cdot \frac{\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1}}{\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1}}$$

$$= \frac{(x+h)^2 - 1 - (x^2 - 1)}{h(\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1})}$$

$$= \frac{x^2 + 2xh + h^2 - 1 - x^2 + 1}{h(\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1})}$$

$$= \frac{h(2x + h)}{h(\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1})}$$

$$= \frac{2x + h}{\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1}}$$

(c) $f(x) = \dfrac{1}{x^2}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$$

$$= \frac{\dfrac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} = \frac{\dfrac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}}{h}$$

$$= \frac{-2xh-h^2}{hx^2(x+h)^2} = \frac{h(-2x-h)}{hx^2(x+h)^2}$$

$$= \frac{-2x-h}{x^2(x+h)^2}$$

(d) $f(x) = \dfrac{x}{1+x^2}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{x+h}{1+(x+h)^2} - \dfrac{x}{1+x^2}}{h}$$

$$= \frac{\dfrac{(x+h)(1+x^2) - x(1+(x+h)^2)}{(1+(x+h)^2)(1+x^2)}}{h}$$

$$= \frac{x + x^3 + h + hx^2 - x - x^3 - 2x^2h - xh^2}{h(1+(x+h)^2)(1+x^2)}$$

$$= \frac{-x^2h - xh^2 + h}{h(1+(x+h)^2)(1+x^2)}$$

$$= \frac{h(-x^2 - xh + 1)}{h(1+(x+h)^2)(1+x^2)}$$

$$= \frac{-x^2 - xh + 1}{(1+(x+h)^2)(1+x^2)}$$

2. The number of pounds (in millions) of lobster caught by Maine commercial fisherman is given by $P(t)$, where t is measured in years. Selected values for $P(t)$ are given in the table.

t	2010	2012	2014	2016	2018	2020	2022
$P(t)$	255.8	318.3	306.9	302.9	285.8	205.1	107.1

Find the average rate of change in pounds of lobsters caught **(i)** from 2016 to 2018; **(ii)** from 2018 to 2020.

Indicate the unites of measure. What does your answers suggest about the change in the number of pounds of lobster caught in recent years?

$$\text{AROC} = \frac{P(2018) - P(2016)}{2018 - 2016} = \frac{285.8 - 302.9}{2} = \frac{-17.1}{2} = -8.55 \ \text{ mil pounds/year}$$

$$\text{AROC} = \frac{P(2020) - P(2018)}{2020 - 2018} = \frac{205.1 - 285.8}{2} = \frac{-80.7}{2} = -40.35 \ \text{ mil pounds/year}$$

The number of pounds of lobsters caught per year is decreasing at a faster rate.

3. A particle moves along a horizontal number line. Its position at time $t \geq 0$ is given by $s(t) = t^2 - 7t + 2$ where t is measured in seconds and s is measured in meters.

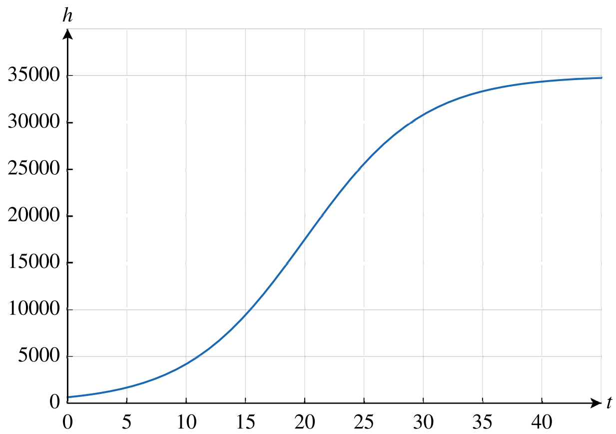
(a) Find the average rate of change in the particle's position from $t = 0$ to $t = 8$ seconds.

$$\frac{s(8) - s(0)}{8 - 0} = \frac{(8^2 - 7 \cdot 8 + 2) - (2)}{8 - 0} = \frac{8}{8} = 1 \text{ meters/sec}$$

(b) Use your answer in part (a) to determine if the particle is to the left or the right of its initial position at time $t = 8$.

Since the average rate of change is 1 (> 0) m/sec, the particle is to the right of its initial position at time $t = 8$.

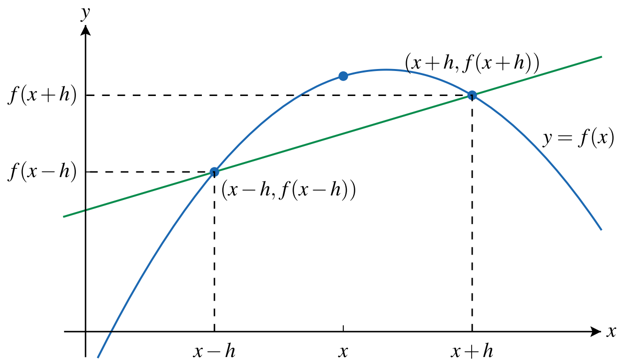
4. The figure shows the graph of the altitude of a plane (h, in feet) from takeoff, $t = 0$ minutes, to $t = 45$ minutes.



Use the graph to determine on which five-minute interval, 0-5, 5-10, 10-15, etc, the average rate of change in height is greatest.

5. For each of the following functions, simplify the expression

$$\frac{f(x+h) - f(x-h)}{2h}$$



(a) $f(x) = 2x + 5$

$$\begin{aligned}\frac{f(x+h) - f(x-h)}{2h} &= \frac{2(x+h) + 5 - (2(x-h) + 5)}{2h} \\ &= \frac{2x + 2h + 5 - 2x + 2h - 5}{2h} \\ &= \frac{4h}{2h} = 2\end{aligned}$$

(b) $f(x) = x^2 + 3x + 4$

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{(x+h)^2 + 3(x+h) + 4 - ((x-h)^2 + 3(x-h) + 4)}{2h}$$

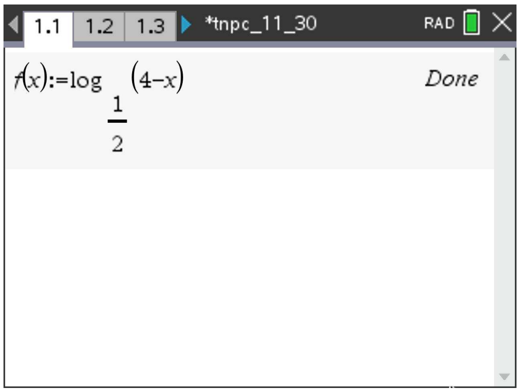
$$= \frac{x^2 + 2xh + h^2 + 3x + 3h + 4 - (x^2 - 2xh + h^2 + 3x - 3h + 4)}{2h}$$

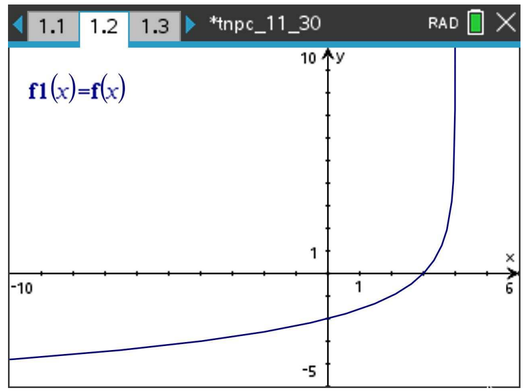
$$= \frac{4xh + 6h}{2h}$$

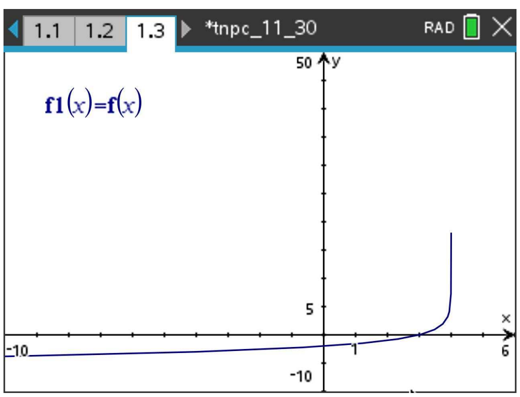
$$= \frac{2h(2x+3)}{2h} = 2x + 3$$

Overtime Problems

1. Use technology to sketch the graph of $y = \log_{1/2}(4 - x)$







2. The value, in millions of dollars, for sales of luxury tile flooring from a distributor is modeled by the function F. The value is expected to increase by 3.2% each quarter of a year. At time $t = 0$, 37 million dollars of sales were processed. If t is measure in years, write an equation for $F(t)$.

Total amount after interest is compounded n times per year:

$$A = P_0\left(1 + \frac{r}{n}\right)^{nt}$$

$A =$ Accumulated amount
$P_0 =$ Original investment amount
$r =$ Interest rate (in decimal form, yearly)
$t =$ Time (years)
$n =$ Number of times interest is compounded in a year

$$A = 37(1 + 0.032)^{4t} = 37(1.032)^{4t}$$