

# Building a Better Pet Run

## Task:

A 4-H Club decided to build a new outdoor run for the local Humane Society. They have 60 feet of chain-link fence. Club members are trying to design the “best” run but cannot visualize how the area of the run changes for different widths. What mathematical model could be used to explain how the area changes?

## Construct and Investigate:

- Construct a line across the top of the Voyage™ 200 screen. Place  $\overline{AB}$ , a 3-cm segment, on the line, and hide the line (Figure 1). Place point  $C$  on  $\overline{AB}$ . Use **Compass** to transfer the length  $\overline{AC}$  to another part of the screen. Draw  $\overline{FG}$ , the radius of the resulting circle. Use segment  $\overline{FG}$  as the width of the rectangular pen. Draw perpendicular lines at both ends of segment  $\overline{FG}$ .
- Use **Compass** to transfer the length  $\overline{CB}$  to  $F$ . Construct a perpendicular line at the point of intersection of the second compass circle and the previously constructed perpendicular line through  $F$  (Figure 1). Using the **Polygon** tool, overlay quadrilateral  $FGHK$ . Hide all construction lines. Measure the perimeter and area of  $FGHK$ , and drag  $C$  on  $\overline{AB}$ . Explain what happens (Figure 2). If 1 cm = 10 feet, explain how this drawing models the pet run problem. Explain the “best” run using your drawing to find its size.
- Using the **Collect Data** option **Define Entry**, select the width, area, and perimeter of  $FGHK$  for data collection. Drag  $C$  on  $\overline{AB}$ , and use  $\blacklozenge$  **D** to collect the data. You can view the data in the **Data/Matrix Editor** or by selecting **Data View** from the **F8** menu. Collect at least 15 data values for  $C$  at various positions on  $\overline{AB}$ .
- Using the **Plot** options of the Voyage 200, draw a scatter plot of the data collected on the fixed perimeter rectangle. Is there a pattern? Use the regression tools of the Voyage 200 to fit a model to the data. Explain algebraically why this model is correct.

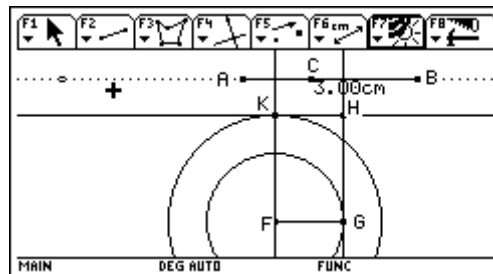


Figure 1

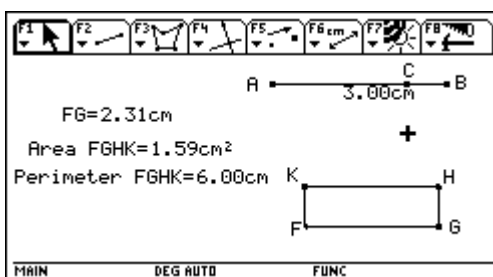


Figure 2

## Explore:

- On the Cabri program menu, select  $\boxed{F8}$ , select **Format**, and press  $\boxed{\text{ENTER}}$ . Select **RECTANGLE** as the Coordinate Axes. Place the origin near the lower left corner of the screen. Use **Measurement Transfer** to transfer the width of the rectangle to the  $x$  axis and the area of the rectangle to the  $y$  axis. Construct lines perpendicular to each axis at the points representing width and area. Place point  $P$  at the intersections of the two perpendicular lines. Hide the lines. Move  $C$ , and explain how point  $P$  moves.
- Use **Trace** to show the path of  $P$  as  $C$  moves. Use **Locus** to show the locus of  $P$  as  $C$  moves. Explain what you found and how it compares with the regression model from part 4 above. Show the algebraic connections of this model to the pet run problem. What would happen if the length of the original line segment is changed? Explain this algebraically by generalizing.

# Teacher's Guide: Building a Better Pet Run

## Construct and Investigate:

- The original line is used to control the position of the segment, so its length can be changed without changing its orientation on the screen. This method makes the last exploration in part 2 of *Explore* easy to accomplish.
- The second **Compass** circle can be placed at either end of  $FG$ ; the construction will still be valid. Because  $AC = FG$ ,  $CB = FK$ ,  $FGHK$  is a rectangle and  $AC + CB = AB$ , the segment  $\overline{AB}$  represents half the perimeter of  $FGHK$ . Dragging  $C$  on  $\overline{AB}$  does not change the sum  $AC + CB$ , only the individual parts. Hence, the rectangle  $FGHK$  has a constant perimeter equal to twice the length of  $\overline{AB}$ . If the length of  $\overline{AB}$  is changed, the perimeter of  $FGHK$  changes, but no matter where  $C$  is located on  $\overline{AB}$ , the perimeter of  $FGHK$  is constant. The same is not true for the area of  $FGHK$ . It varies between 0 and  $2.25 \text{ cm}^2$  as  $C$  is moved along  $\overline{AB}$ . As the lengths of  $\overline{FG}$  and  $\overline{FK}$  are changed, the product  $FG * FK$  changes, even though the sum remains constant.

The length of  $10 * FG$  represents all the different widths of the pet run, from 0 to 30 feet. The area of  $FGHK * 100$  represents the number of square feet enclosed in the pet run, from 0 to  $225 \text{ ft}^2$ . The definition of "best" pet run could be the largest area enclosed, or it could involve a long narrow run that has space for the pets to really run. The maximum pen area is  $15 \text{ ft}^2$ . Select the measurements, and type  $\oplus$  to show the value with more digits of precision (Figure 3).

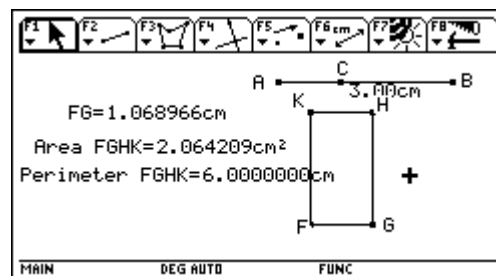


Figure 3

- Collect data for the width, area, and perimeter of rectangle  $FGHK$  for  $C$  in various positions across  $\overline{AB}$  (Figure 4). Changing the width of the rectangle changes the area, but the perimeter remains constant. Draw a scatter plot of this data with the width on the  $x$  axis and the area on the  $y$  axis. The domain of the geometric model is  $0 \text{ cm} \leq x \leq 3 \text{ cm}$ . The domain of the pet run problem is  $0 \text{ ft.} \leq x \leq 30 \text{ ft.}$
- The scatter plot of this data shows a parabolic pattern (Figure 5). Quadratic regression gives the equation of a model:

$$y = -x^2 + 3x.$$

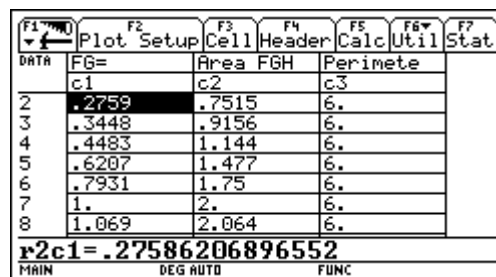


Figure 4

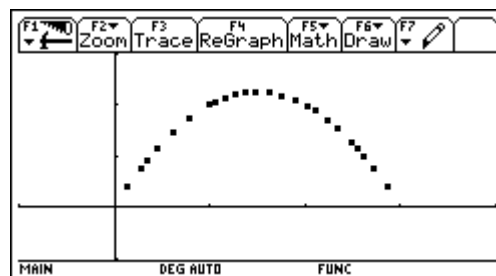


Figure 5

## Teacher's Guide: Building a Better Pet Run (Cont.)

The graph of the model over the data shows that the graph passes through every data point (Figure 6).

From your exploration, you know that the perimeter of the pet run is 60 feet. This can be expressed algebraically as:

$$2l + 2w = 60, \text{ or } l + w = 30, \text{ and then } l = 30 - w.$$

By substituting into the general rectangular area formula  $A = lw$ , you get the expression:

$$A = (30 - w)w = -w^2 + 30w.$$

Because your geometric model is scaled to 1 cm = 10 feet, the same relationship between length, width, and area exists, but it is adjusted for the **Scale Factor** in the following manner:

$$A = (3 - w)w = -w^2 + 3w.$$

This function is the same one you derived as a quadratic regression model that exactly fits the data. You can find the maximum value for this function graphically by using **Zoom-In** or the built-in tools of the Voyage™ 200 as shown in Figure 7. When  $x = 1.5$ ,  $y = 2.25$ . For the pet run, when the width is 15 feet, the area of the pen is 225 ft<sup>2</sup>. This dimension represents a square because the width is also 15 feet.

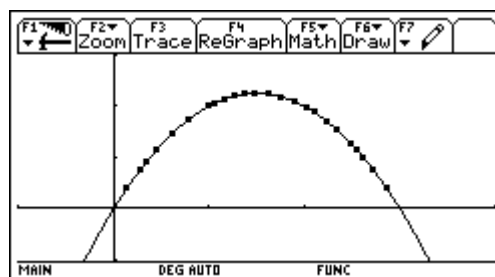


Figure 6

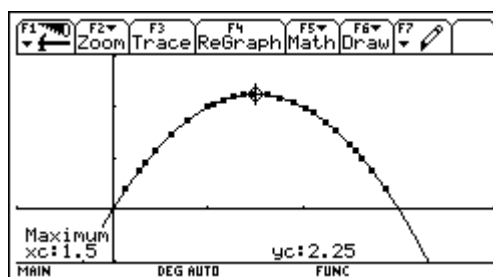


Figure 7

## Teacher's Guide: Building a Better Pet Run (Cont.)

### Explore:

1. **Measurement Transfer** allows the measurements of width and area to be transferred to the  $x$  and  $y$  axes, respectively. The coordinates of point  $P$ , located at the intersections of the lines perpendicular to the axes through these two points, represent a width and corresponding area for the rectangle (Figure 8). As the width changes when you drag  $C$ , the area of the rectangle changes from zero, to a maximum value and then back to zero. Point  $P$  is on the graph of  $y = -x^2 + 3x$ .
2. Trace point  $P$  to show the same type of information as the scatter plot of the data (Figure 9).

The locus of point  $P$ , as  $C$  is moved along  $\overline{AB}$ , is the portion of the parabola  $y = -x^2 + 3x$ , representing the problem situation with the domain  $0 \leq x \leq 3$  (Figure 10). Test this conjecture by taking any point on the locus represented by the coordinates of  $P$  and evaluating the function  $y = -x^2 + 3x$  at the chosen  $x$ . As shown in Figure 10,  $y = -(2.22)^2 + 3(2.22) = 1.73$ . Because the geometric model is scaled to  $1 \text{ cm} = 10 \text{ feet}$ , the function for the area of the pet run is given by the model  $y = -x^2 + 30x$ . For a pet run 22.2 feet wide,  $A(22.2) = -(22.2)^2 + 30(22.2) = 173.2 \text{ ft}^2$ .

Changing the length of the original segment  $\overline{AB}$  redefines the basic conditions of the problem. Segment  $\overline{AB}$  represents half the perimeter of the rectangle  $FGHK$ . If the length of  $\overline{AB}$  is changed to 4 cm, then the fixed perimeter of  $FGHK$  is 8 cm. The area of  $FGHK$  changes between zero and  $4 \text{ cm}^2$ . The function describing this width-area relationship is  $y = -x^2 + 4x$  because the length of the rectangle is given by  $l = 4 - w$ . In Figure 11, for the point  $(2.93, 3.13)$ ,  $A = -(2.93)^2 + 4(2.93) = 3.13$ .

In general, the width of the rectangle is given by the expression  $l = c - w$ , where  $c$  is the length of  $\overline{AB}$ , or half the perimeter of the rectangle  $FGHK$ . Therefore, the general function representing the area of any rectangle with a perimeter of  $2c$  is  $A(w) = -w^2 + cw$ .

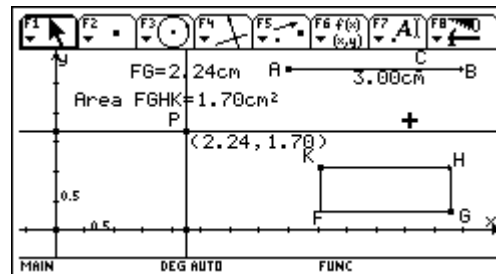


Figure 8

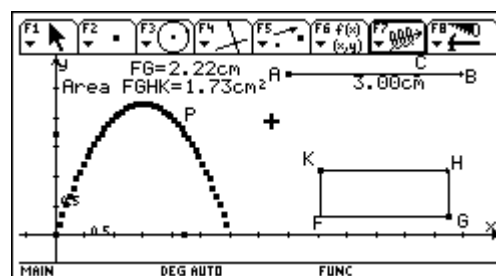


Figure 9

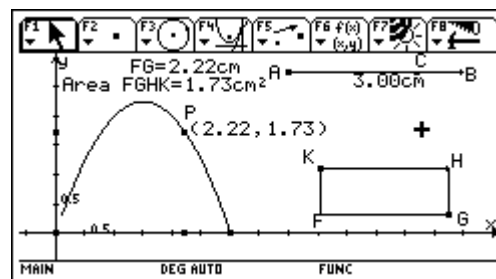


Figure 10

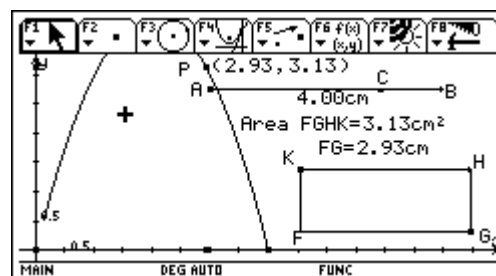


Figure 11