Learning Exponentially

Teachers Notes & Answers

7 8 9 10 **11** 12

TI-Nspire™ | Investigation | Student | 50 min

Introduction

What happens to the graph of an exponential function when it is dilated, reflected or translated? What combinations of these transformations are homomorphic? In this investigation you will use TI-nspire technology to visually and numerically explore these transformations.

Teachers Notes:

Transformations of exponential functions can be confusing for some students as some transformations can be replicated with parameters in different locations. For example: $f(x)=4\times 2^x$ and $g(x)=2^{x+2}$ produce the same function. The dilation by 4 in $f(x)$ is the same as the translation in $g(x)$. This activity works progressively through transformations of an exponential function, highlighting along the journey that some transformations are the same, homomorphic. To help this concept stick, students are reminded that many jokes rely on 'homophones'.

Tip

Q. Why can't Shetland ponies speak?

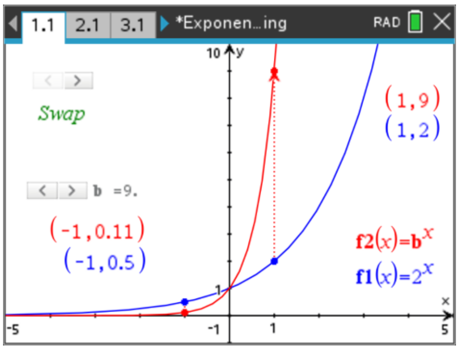
A. They're always a little *hoarse*.

Jokes often rely on '**homophones**', words that spelt differently but pronounced the same. In mathematics, **homomorphic** expressions may look different but produce the same result.

Part A: Exploring $y=b^x$

Open the TI-Nspire file: Learning Exponentially.
Navigate to page 1.1
There are two functions in this graph application:

$$f_1(x)=2^x \text{ and } f_2(x)=b^x$$



The purpose of Part A is to explore what happens to the points on the curve as the value of *b* is changed. Values for *b* range from 2 to 10.
Two points are identified specifically to help understand what is happening.
The movement of these points may be considered in the x or y direction.
Use the 'swap' slider to toggle the direction.

Question: 1

As the base (*b*) is changed from 2 through to 10:

a) Describe what happens to the points $(1,2)$ and $\left(-1,\frac{1}{2}\right)$ by consideration of the ordinate (y coordinate).

For the point (1, 2), as *b* increases the ordinate increases. The reverse is true for the point (-1, ½), as *b* increases ordinate decreases.

Teacher Notes:

The purpose of this question is for students to consider individual points on the function and see that referencing their respective movements in the y direction cannot be described consistently for all points. A

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useful analogy for students is to imagine the graph has been drawn on an elasticised piece of graph paper. You can stretch the paper in a direction 'parallel to the y axis' which is the same as 'away from the x axis' and consider how all the points would move. Similarly you could stretch the paper in a direction 'parallel to the x axis" which is the same as 'away from the y axis.' This analogy further builds on the understanding that all points on the graph are being transformed the same way. So, to produce the transformation resulting in varying *b* in this part of the investigation, the 'stretching' could not be done parallel to the y axis or away from the x axis as some points (those where x < 0) are actually moving closer to the x axis while others (x > 0) are moving away. Question (c) is designed to further enhance and build upon this understanding.

b) Which points, if any, are invariant? (Explain)

The point (0, 1) is invariant. (Does not change)

Teacher Notes:
This is consistent with dilation *from the y axis* or *parallel with the x axis* as the abscissa (x coordinate) is zero.

c) Describe what happens to the points $(1,2)$ and $\left(-1,\frac{1}{2}\right)$ by consideration of the abscissa (x coordinate).

For the point (1, 2), as *b* increases the abscissa decreases. The reverse is true for the point (-1, ½), as *b* increases the abscissa increases, however, all points are moving 'towards' the y axis.

Teacher Notes:
The purpose of this question is for students to understand that all points on the curve could be visualised as moving towards the y axis. This collective movement means that we can describe the transformation as a dilation 'towards' the y axis.
If students are still having difficulty comprehending the description of the dilation, consider the Geometric Transformation activities available on the Texas Instruments Australia website.

d) Describe what happens to the *shape* of the graph as *b* is changed.

The graph is dilated towards the y axis (parallel to the x axis) as *b* increases.

e) Identify the asymptote(s) for each of the graphs.

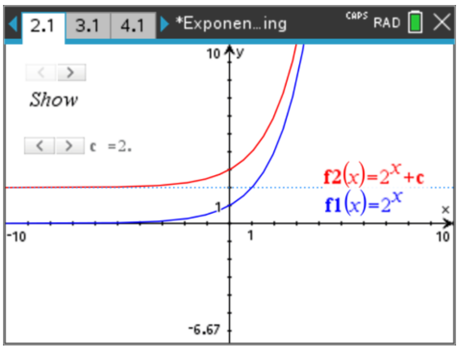
For each graph the equation of the asymptote remains the same: $y=0$ since there are no translations parallel to the y axis.

Part B: Exploring $y=2^x+c$

Navigate to page 2.1
There are two functions in this graph application:

$$f_1(x)=2^x \text{ and } f_2(x)=2^x+c$$

The slider can be used to change the value of *c*.
Try values for *c*: ⁻5 through to 5.
The show / hide slider can be used to help visualise how individual points are moving.



Question: 2

As the value of *c* is changed from -5 through to 5:

a) Describe what happens to the *shape* and *location* of the graph.

The *shape* of the graph is not changed (no dilations), however, the location of the graph has been changed, the graph has been translated parallel to the y axis by the quantity *c* from the x axis.
The ordinate of every point on the curve is changed by the same quantity, *c*, this can be seen when individual points are revealed using the show / hide slider.

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b) Identify the asymptote(s) for each of the graphs.

In general, the equation to the asymptote of this graph is: y = c

Teacher Notes:

The TI-Nspire software does not automatically show the graph of an asymptote. The equation has been entered into the graph page and the label has been hidden.

c) For what range of values for *c* does $f_2(x)=0$ have a solution?

If c < 0 then the graph has solutions for $f_2(x)=0$ when $y=\log_2(-c)$.

Teacher Notes:

The question only requires students identify the range of values for c, not the actual solutions as the solutions presumes that students have already started solving equations with exponentials.

Part C: Exploring $y=2^{x-h}$

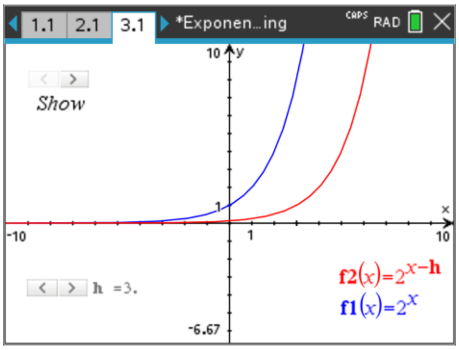
Navigate to page 3.1

There are two functions in this graph application:

$$f_1(x)=2^x \text{ and } f_2(x)=2^{x-h}$$

The slider can be used to change the value of *h*.

Try values for *h*: ⁻5 through to 5.



Question: 3

As the values of *h* is changed from -5 through to 5:

a) Describe what happens to the *shape* and *location* of the graph.

The *shape* of the graph is not changed (no dilations), however, the location of the graph has changed; the graph has been translated parallel to the x axis by the quantity *h* from the y axis.
The abscissa of every point on the curve is changed by the same quantity, 'h', this can be seen when individual points are revealed using the show / hide slider.

b) Identify the asymptote(s) for each of the graphs.

The asymptote equations are the same for all of these graphs as the translation parallel to the x axis does not change the equation to the horizontal asymptote: y = 0

Part D: Exploring $y=2^{mx}$

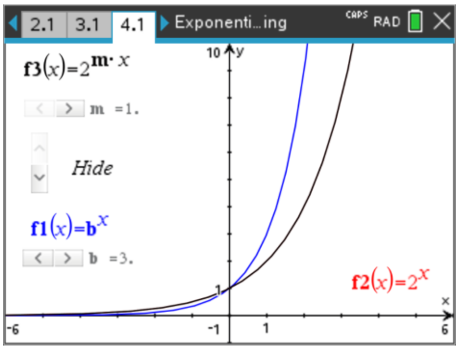
Navigate to page 4.1

There are three functions in this graph application:

$$f_1(x)=b^x \text{ , } f_2(x)=2^x \text{ and } f_3(x)=2^{mx}$$

The show/hide slider can be used to reveal the graph of $f_1(x)$.

Sliders can be used to change the values of *m* and *b* once the additional graph has been revealed.



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Teacher Notes

The purpose of this section is to highlight the homomorphic representations for these transformations. A base of 2 was selected so that students should recognise that y = 2^{2x} will produce the same graph as y = 4^x and see this connection by use of index laws. Students may extend this to $b = 2^m$ and $m = \log_2(b)$ but should also recognise some limitations of this equivalence ($b > 0$).

Question: 4

As the values of *m* is changed from 1 through to 4 in the graph of $f(x) = 2^{mx}$:

a) Describe what happens to the *shape* and *location* of the graph.

The location of the graph is unchanged (no translations) but the shape is dilated 'towards' the y axis or 'parallel to the x axis'. The dilation is by a factor of 1/m parallel to the x axis or toward the y axis.

b) Are there any invariant points on the graph of: $f(x) = 2^{mx}$?

The point (0, 1) is invariant.

There is a show / hide button on the Graph [Page 4.1]. Use this to reveal another graph that has similar form, one that has been explored previously.

Question: 5

Explore values of *m* and *b* for the graphs of: $f(x) = 2^{mx}$ and $f(x) = b^x$, determine values for which the graphs are homomorphic (same) and explain the outcome.

Answers will vary: For the values available from the sliders students should see that for m = 3 then b = 8 produces the same transformation, similarly for m = 2 then b = 4 and m = 4 and b = 16.
In general $b = 2^m$ or $m = \log_2(b)$. Note that these results are true for b > 0.

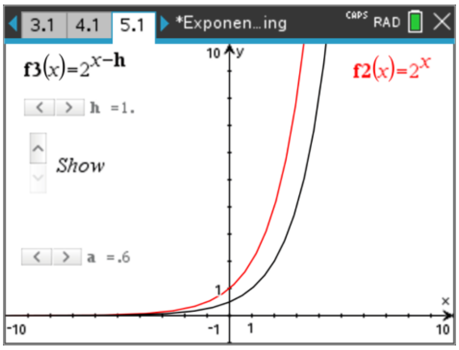
Part E: Exploring $y = a \cdot 2^x$

Navigate to page 5.1
There are three functions in this graph application:

$$f_1(x) = a \cdot 2^x,\ f_2(x) = 2^x \text{ and } f_3(x) = 2^{x-h}$$

The show/hide slider can be used to reveal the graph of $f_1(x)$.

Sliders can be used to change the values of *a* and *h* once the additional graph has been revealed.



Question: 6

Explore values of *a* and *h* for the graphs of: $f(x) = 2^{x-h}$ and $f(x) = a \cdot 2^x$, determine values for which the graphs are homomorphic (same), explain the outcome and identify limitations for the comparison.

Answers will vary: Restricting solutions to only those provided through exploration would produce:

a = ½ and h = 1

a = 1 and h = 0 (trivial)

a = 2 and h = -1

a = 4 and h = -2

Using index laws students should see that $2^{x-h} = \frac{2^x}{2^h}$ therefore, the graphs are the same when $a = \frac{1}{2^h}$.

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Students that reach this point should also realise that restrictions are imposed for $f(x)=2^{x-h}$ compared with $g(x)=a\times 2^{x}$ since $a=\frac{1}{2^{h}}$ restricts $a>0$ and that if $a<0$ a reflection in the x axis results.

Question: 7

Compare the transformations of $f(x)=2^{x-h}+c$ with those for $g(x)=(x-h)^{2}+c$.

Transformations involving c:

For both exponential and quadratic functions the parameter *c* produces a translation of c units parallel to the y axis and away from the x axis. The transformation produced by c in $f(x)$ and $g(x)$ has the same effect.

Transformations involving *h*:

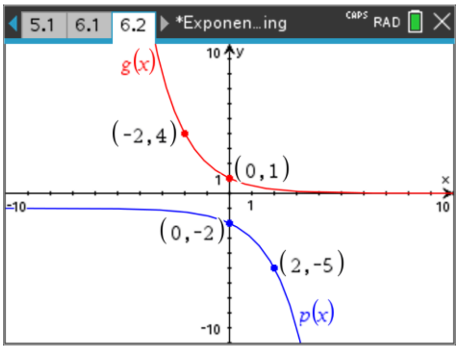
For both exponential and quadratic functions, the parameter *h* provides a translation of *h* units parallel to the x axis. The transformation produced by *h* in $f(x)$ and $g(x)$ has the same effect.

However, since $2^{x-h}=2^{x}\times 2^{-h}=\frac{1}{2^{h}}2^{x}$ then the translation produced by *h* can also be read as a dilation.

Part F: Combining Transformations

Navigate to page 6.1

This problem contains a blank graph application for you to use with the aim of matching the two graphs shown opposite. Each graph has been created using just one of the transformation parameters explored so far. The values for the parameters however are outside those explored.



Question: 8

The graph of $g(x)$ is of the form: $g(x)=a\cdot 4^{mx}+c$. Identify possible values for the parameters such that the function passes through (-2, 4) and (0, 1) and has an asymptote at $y=0$.

Asymptote at y = 0 indicates that c = 0

Students could solve the problem using simultaneous equations; however simplistic values have been selected for the parameters so students may arrive at the solution using 'guess and check'.

$g(0)\Rightarrow a\cdot 4^{m\times 0}+0=1$ and $g(-2)\Rightarrow a\cdot 4^{-2m}+0=4$ therefore *a* = 1 and m = -½

Question: 9

The graph of $p(x)$ is of the form: $p(x)=a\cdot 8^{mx}+c$. Identify values for the parameters such that the function passes through (2, -5) and (0, -2) and an asymptote at y = -1.

Asymptote at y = -1 indicates that c = -1

Students could solve the problem using simultaneous equations; however simplistic values have been selected for the parameters so students may arrive at the solution using 'guess and check'.

$g(0)\Rightarrow a\cdot 8^{m\times 0}-1=-2$ and $g(-2)\Rightarrow a\cdot 8^{-2m}-1=-5$ therefore *a* = -1 and m = 1/3

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