Zeros of a Cubic

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Abstract: This activity involves some of the prerequisites of calculus relating to functions and equations. It also contains an application of differentiation. It introduces students to an interesting property of cubics and a method of proving that property using the TI-89 scripts. They then use the symbolic capacity of their calculator to generalize upon specific results.

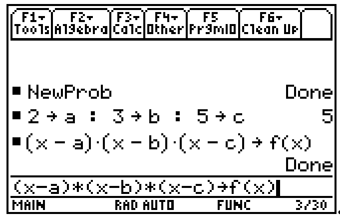
NCTM Principles and Standards:
Algebra standards
a) Understand patterns, relations, and functions
b) generalize patterns using explicitly defined and recursively defined functions;
c) analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
d) use symbolic algebra to represent and explain mathematical relationships;
e) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
f) draw reasonable conclusions about a situation being modeled.

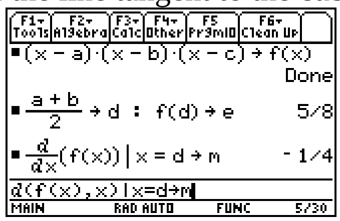
Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.

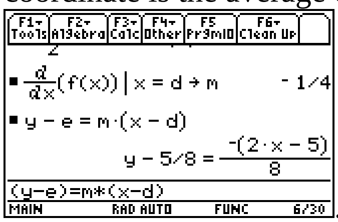
Reasoning and Proof Standard
a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

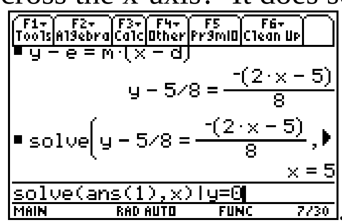
Key topic: Application of differentiation. Prerequisites of calculus relating to functions and equations. Scripts, formal proofs.

Degree of Difficulty: Elementary
Needed Materials: TI-89 calculator
Situation: Cubic polynomials have many interesting properties. In this activity we'll investigate one of them with the aid of the TI-89 calculator. What is the relationship between the x-coordinates of the three zeros of a cubic?

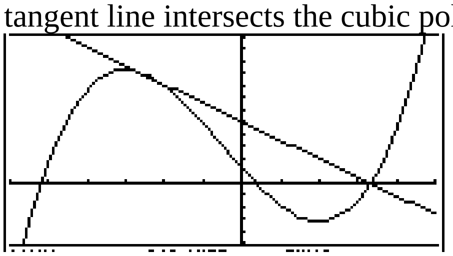
Choose three distinct and arbitrary values for the zeros of the cubic. Form the equation of the cubic and store this as f(x). 

. Now take any two of these zeros and find the equation of the line tangent to the cubic at the point whose x-coordinate is the average of the zeros. 

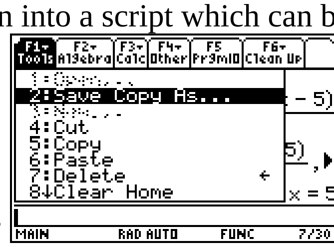


. Where does this tangent line cross the x-axis? It does so when y=0, so we'll solve this equation for x when y = 0: 

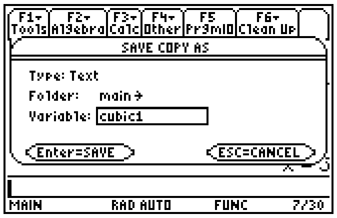
. The tangent line intersects the cubic polynomial at the third zero!


Is this always true? Consider a different set of zeros and try this process again. To find out whether this always works, we'll use what we've done so far to create a script:

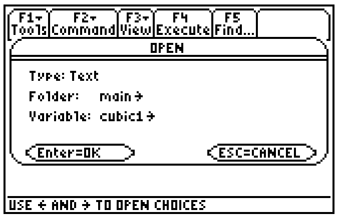
We'll turn what we've written into a script which can be followed for any cubic:

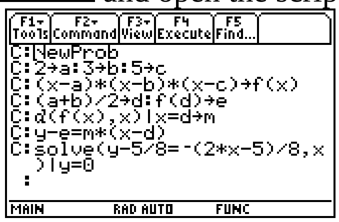
Press F1 and choose "Save Copy As" 

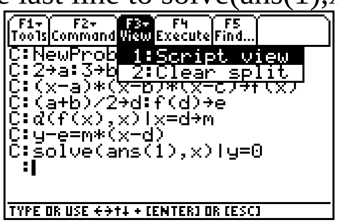
I named this script

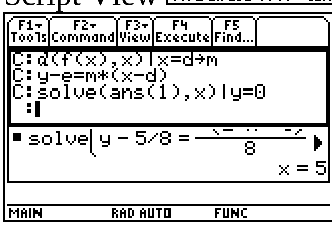
"cubic1" 

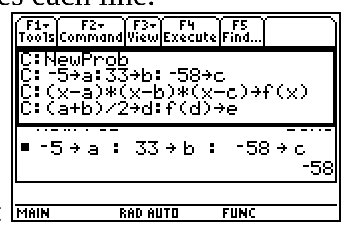
From the application menu [APPS] choose 8:Text Editor: 

and open the script named cubic1: 

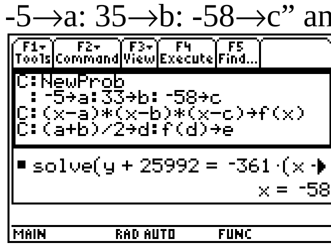


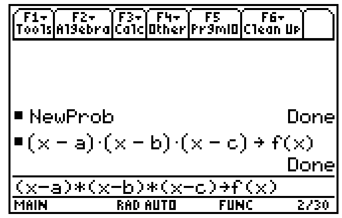
This script contains the commands that were originally typed in. Before we can play this script for other choices of a, b, and c, change the last line to solve(ans(1),x)|y=0. Now press [F3]to change the view to A: Script View 

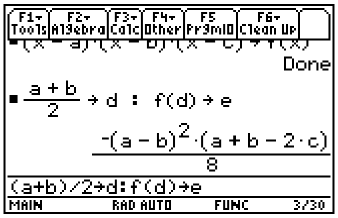
Pressing [F4] executes each line. 

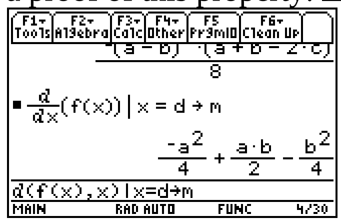
Change the values of a, b, and c: 

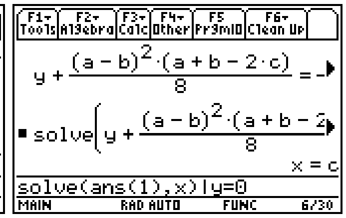
and continue pressing [F4] to see that the property is true with these choices.

Is it always true? Go back to the line "C: -5→a: 35→b: -58→c" and choose 4:Clear Command from the [F2] Command menu: 

which will erase the C: in the line "C: -5→a: 35→b: -58→c". Now run the script again and observe that the calculator creates a proof of this property: 







Scripts can be very useful in proving properties. Here we showed that the tangent line to any cubic at the point whose x-coordinate is the average of two zeros always passes through the third zero. There is one problem. What if the cubic doesn't have three distinct real zeros? Go back and look at the proof we generated. It is still valid if any two, or even all three of the zeros are the same. But what if two of the zeros are complex?