

Chapter 8: Numerical Methods and Series

- Using Euler's method with **tStep** = 0.2, you get the following final values at $t = 25$ (with approximate times for graphing in parentheses):

Estep = 1 gives $Q1(25) = 1.88238882969$ and $Q2(25) = -0.63721596765$ (in 15 sec);

Estep = 2 gives $Q1(25) = -2.19785946694$ and $Q2(25) = -0.186053257724$ (in 22 sec);

Estep = 3 gives $Q1(25) = -1.91828275908$ and $Q2(25) = 0.584942960168$ (in 36 sec);

Estep = 4 gives $Q1(25) = -1.68685411767$ and $Q2(25) = 0.703272382687$ (in 38 sec);

Estep = 5 gives $Q1(25) = -1.53553509458$ and $Q2(25) = 0.785156224644$ (in 45 sec);

Estep = 10 gives $Q1(25) = -1.19596142877$ and $Q2(25) = 1.00637048537$ (in 103 sec).

$$\begin{aligned}y(t) &\approx \frac{1}{3} - \frac{10}{9}t + \frac{1}{2}\left(\frac{68}{9}\right)t^2 + \frac{1}{6}\left(-\frac{250}{9}\right)t^3 + \frac{1}{24}\left(\frac{1250}{9}\right)t^4 + \frac{1}{120}\left(-\frac{6250}{9}\right)t^5 \\ &= \frac{1}{3} - \frac{10}{9}t + \frac{34}{9}t^2 - \frac{125}{27}t^3 + \frac{625}{108}t^4 - \frac{625}{108}t^5\end{aligned}$$

Example 2 had initial conditions at $t = 0$, while here you have initial conditions at $t = 1$. Thus the only changes needed in the program are to change the sixth line to be:

```
(eval (tMin+tStep)) → LE,
```

the tenth line to be:

```
aug(LE, eval (tMin+tStep)) → LE,
```

and the thirteenth line to be:

```
LE-y1(tMin+HH) → LE.
```

Also needed are the appropriate differential equation and new window settings. Following the steps of Example 2, you find that the local errors are approximately -0.5 times h^2 and the global errors are approximately -0.0987 times h . Thus in order to get accuracy of $1E-7$, you would need $h < 1.013E-6$ or at least 987, 167 Euler steps.

2. Check the equations given for the general three-stage method to be h^3 .

$$w_1 + w_2 + w_3 = \frac{1}{9} + \frac{1}{3} + \frac{4}{9} = 1, \quad w_2 \alpha_2 + w_3 \alpha_3 = \frac{1}{3} \left(\frac{1}{2}\right) + \frac{4}{9} \left(\frac{3}{4}\right) = \frac{1}{2},$$

$$w_2 \alpha_2^2 + w_3 \alpha_3^2 = \frac{1}{3} \left(\frac{1}{2}\right)^2 + \frac{4}{9} \left(\frac{3}{4}\right)^2 = \frac{1}{3}, \quad \beta_{2,1} = \frac{1}{2} = \alpha_2 = \frac{1}{2},$$

$$\beta_{3,1} + \beta_{3,2} = 0 + \frac{3}{4} = \alpha_3 = \frac{3}{4}.$$

3. Numerically, the final computed values for Euler's method applied to this problem are

```
y(1) Å 0.653571563519 (h = 0.1),
y(1) Å 0.673839785569 (h = 0.05),
y(1) Å 0.691258668286 (h = 0.005),
y(1) Å 0.692392889375 (h = 0.002).
```

Using the internal **RK** methods and **tStep** = 0.01, you get

```
y(1) Å 0.693103519183 (difTol = 1E-2),
y(1) Å 0.693103519183 (difTol = 1E-3),
y(1) Å 0.693145150003 (difTol = 1E-4).
```

4. Replace RK3 by an appropriate RK4 used as a subprogram. Then for the differential equation in Example 4, you get local errors approximately $7E-4$ times h^5 and global errors approximately -0.0121 times h^4 . To get an accuracy of $1E-7$, you would need to choose $h < 0.0536$ or at least 19 of these RK4 steps. While you do one-third more work per step (computing four stages rather than three), this still improves on the RK3 method implemented in Example 4, which required at least 86 steps. The best comparison would be that $(19)(4) = 76$ stages using RK4 will achieve the same accuracy as $(86)(3) = 258$ stages using RK3, or 6.9 million stages (steps) using Euler's method.
5. Even for relatively large difTol, the computation goes slowly as it tries to keep errors under control. Much of the slope field has nearly vertical line segments. You get
- ```
Q1(10) Å -0.852156829289 with difTol = 1E-2,
Q1(10) Å -0.849474326792 with difTol = 1E-3,
Q1(10) Å -0.849679432765 with difTol = 1E-4,
Q1(10) Å -0.849630721399 with difTol = 1E-5.
```