

Activity 7

Modeling Exponential Decay with a Look at Asymptotes

Objectives

- Define parameters A and B in the exponential equation $y = ab^x$
- Understand asymptotes, particularly non-zero asymptotes
- Identify the non-zero asymptote form of an exponential function, $y = ab^x + c$
- Practice using the Transformation Graphing application as a modeling tool
- Explore a situation in which a continuous model is used for a discrete experiment

Introduction

In the previous activity, you started your study of the exponential function, looking at models of exponential growth and decay and approximating models of exponential growth. In this activity, you will approximate models of exponential decay and will learn about asymptotes.

You should complete Activity 6 before you begin this activity.

Modeling the Experiment: Casting Out Sixes

For this activity, you need 36 number cubes (dice). Perform the following experiment. Before each roll, record the number of rolls that have taken place and the number of dice left.

Experiment Steps

1. In the table below the original number of dice in this set (36) has been recorded.
2. Roll these dice.
3. Remove from the set any die that shows a 6 on its top face.
4. Count the remaining dice in the set and record that information in your table:

Number of rolls	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of dice left	36															

Using only the dice left, repeat steps 2–4 until no die remains or you have rolled 15 times, whichever occurs first.

Data Analysis

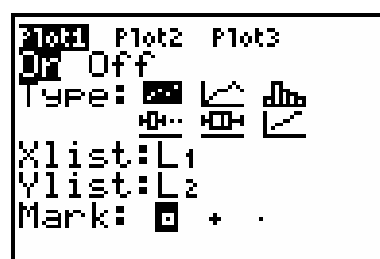
1. Press **[STAT]** 1 to select **1:Edit**. Enter the values from the table into two empty lists in the stat list editor. Clear two lists if no lists are empty.

Note: To clear a list, use the arrow keys to highlight the list name, and then press **[CLEAR]** **[ENTER]**.

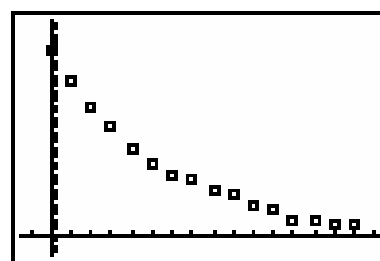
L1	L2	L3	3
0	36		
1	30		
2	25		
3	21		
4	17		
5	14		
6	12		

L3(1)=

2. Press **[Y=]** and clear all functions.
3. To display the plot, press **[2nd]** **[STAT PLOT]** 1 to select **1:Plot1**. Turn the plot on, and select the plot menu settings as shown.



4. Press **[ZOOM]** 9 to select **9:ZoomStat** and display the plot.



What kind of model does the plot suggest? What model does the problem situation lend itself to?

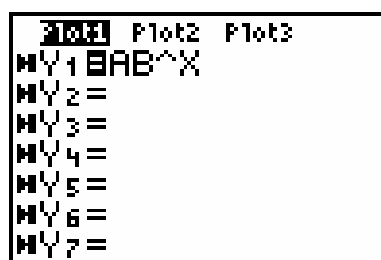
5. Press **[APPS]** and select **Transfrm**. Press any key (except **[2nd]** or **[ALPHA]**) to start the Transformation Graphing App.



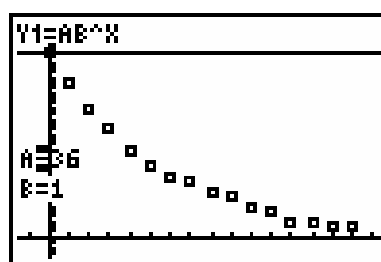
Note: If you do not see the screen shown, select **Continue**.



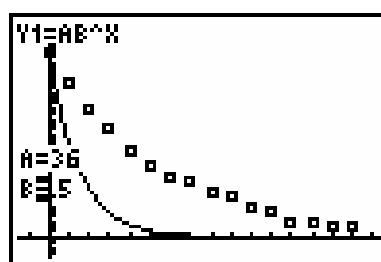
6. a. In **Func** mode, press $\boxed{Y=}$ to display the **Y=** editor. Enter the general form of the exponential function, $y = ab^x$. Press $\boxed{\text{ALPHA}} \boxed{A} \boxed{\text{ALPHA}} \boxed{B} \boxed{\wedge} \boxed{X, T, \theta, n}$.
- If **Play-Pause mode** ($\boxed{>||}$) is not selected at the left of **Y1**, press $\boxed{\leftarrow}$ until the cursor is over the symbol, and then press $\boxed{\text{ENTER}}$ until the correct symbol is selected.
- b. Press $\boxed{\text{GRAPH}}$ to redraw the plot and the graph.



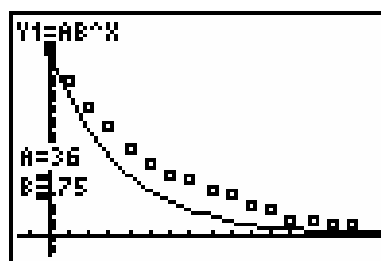
7. Select an appropriate value for A .
- If you started with 36 dice, then the data point $(0, 36)$ (rolls = 0, dice left = 36) is appropriate. The value of A in an exponential function is the value of the dependent variable (in this case the number of dice) when the independent variable (number of rolls) is 0.
- Set $A=36$ and press $\boxed{\text{ENTER}}$.



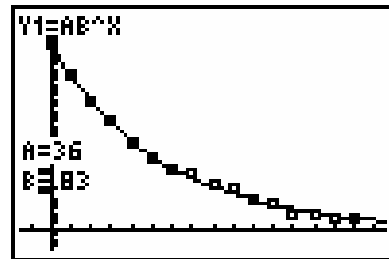
8. Because the data appear to be an example of decay, B must be in the region $0 < B < 1$. To start, set $B=.5$, the middle value in the interval.



9. As you can see, the data do not seem to be decaying as quickly as the model. Therefore, B must be greater than 0.5, in the region $0.5 < B < 1$. As a second guess set $B=.75$, the middle of the revised interval.



10. a. Evaluate the model again. If you think you can get a better model, continue until you believe you have as good a model as you can find. Find the value of B to two decimal places.



Through repeated, systematic guess-and-check, you found a reasonable model for the sample data that is $y = 36(0.83)^x$.

- b. The model has $B = 0.83$. What does this value represent with respect to the problem? Explain.

Notice that the sample data never reached zero. An exponential decay model never reaches 0, but tends to zero, making it **asymptotic**. In the actual experiment presented, however, the data could reach zero.

To find out when there should be no dice left, determine when $y < 0.5$. The assumption is that as long as $y > 0.5$, it rounds to 1, and there will be dice left. When it rounds to less than 1, you will assume that no dice are left.

You will use the table feature of the graphing handheld to determine when $y < 0.5$.

11. Press 2nd [TBLSET] to access the **TABLE SETUP Menu**. Enter the settings shown.

TABLE SETUP	
TblStart=0	
ΔTbl=1	
Indent: [Auto]	Ask
Depend: [Auto]	Ask

12. Press 2nd [TABLE], and then press \downarrow to move down until a value less than 0.5 appears in the **Y1** column.

The model predicts that it will take approximately 23 rolls to remove all of the dice.

X	Y1	
17	1.5158	
18	1.2581	
19	1.0442	
20	.86669	
21	.71935	
22	.59706	
23	.49556	

X=23

An exponential decay model never reaches zero, but this experiment actually does, so this model is reasonable for the situation. Some data sets, like this one, are said to be **pseudo-asymptotic**.

Note: The experiment is a discrete data set, but the exponential function is a continuous model. This fact should be taken into account when the model is used.

Modeling the Experiment: Casting Out Sixes with the Special Number Cubes

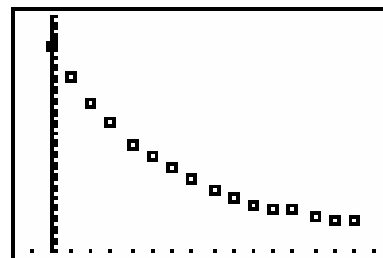
Repeat the Casting Out Sixes experiment with 36 “special” number cubes, available from your teacher. Record your results in the table below.

Number of rolls	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of dice left	36															

Data Analysis

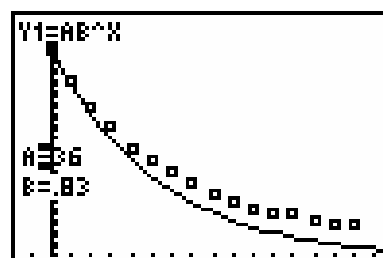
Note: Keystrokes that are the same as were used in the first part of this activity will not be repeated.

1. Plot the data.



What kind of model do the data show? How do these data differ from those in the first experiment?

2. Activate the the Transformation Graphing App, and enter the exponential model, $y = ab^x$. Press **GRAPH**. The last values entered for A and B were from the previous dice problem. The two problems appear the same; thus, these values should be reasonable.



All exponential decay curves are asymptotic, that is, they tend to zero, but the data and the proposed model in this experiment do not appear to have the same asymptote as that of the previous experiment. The asymptote for the data appears to be larger than the $y = 0$ from the previous experiment.

There are two possible ways to improve the model: either try different values for B , or try asymptotes different from $y = 0$.

Because the dice were all six-sided, the value $B = 0.83$ should still be reasonable. So the problem appears to be the wrong asymptote. You will change the model to include an asymptote other than $y = 0$.

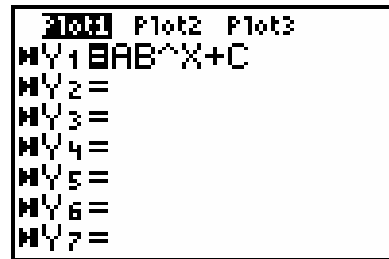
The equation for an exponential model with an asymptote other than $y = 0$ is $y = ab^x + c$ where C is the value of the asymptote.

B remains the growth or decay factor.

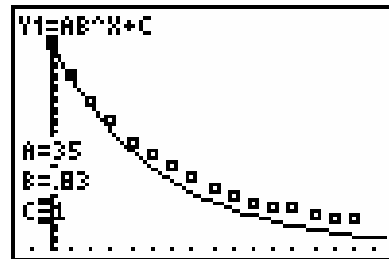
A is the starting point after adjustments for a non-zero asymptote. But it is no longer the y -intercept. The y -intercept is now $A + C$. When $x = 0$ the equation becomes $y = A + C$, ($B^0 = 1$ for all values of B).

For this example $A + C = 36$, because you started with 36 dice.

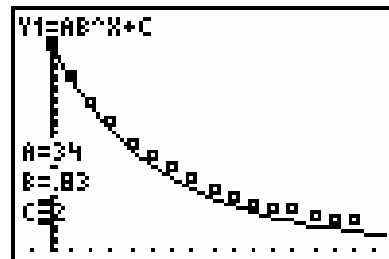
3. Revise your model to $y = ab^x + c$. Try combinations of A and C where $A + C = 36$.



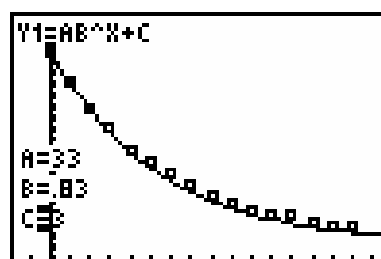
$A = 35 \quad C = 1$



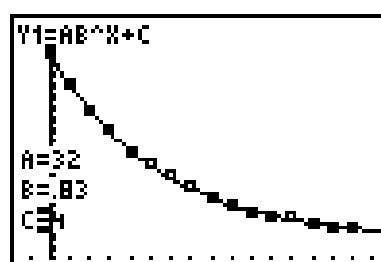
$A = 34 \quad C = 2$



$A = 33 \quad C = 3$



$A = 32 \quad C = 4$



Do any of the revisions appear to produce a reasonable model? Which one?

In this example, the asymptote will be $y = 4$. This means that no matter how many times the dice are rolled, there will always be 4 dice remaining.

- Use the table feature of your graphing handheld to determine when the last die containing a 6 was removed.

A Method for Visualizing the Asymptote

- With the graph and stat plot showing, press 2nd DRAW 3 to select **3:Horizontal**. Press \downarrow to move the horizontal line downward until it appears to be the asymptote. This method will give a rough idea of the location of the asymptote. You could have used this method to get a starting value for C when you started to develop your model.



Note: Horizontal asymptotic behavior is an end behavior. With exponential decay it is best observed with large values of x .

How do you explain the presence of a non-zero asymptote in this experiment?

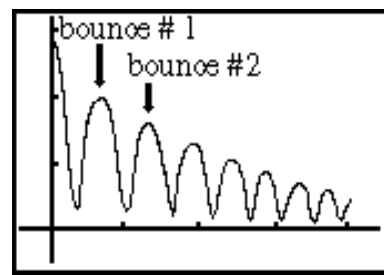
Student Worksheet

Name _____

Date _____

The data for this activity comes from the data that was collected in Activity 4. If you saved the Ball Bounce data in Activity 4, reload it now. If the data was not saved, redo the Ball Bounce data collection.

The data set formed by the number of the bounces and the maximum height of the ball for each bounce can be modeled with an exponential function. Trace along the data set, and record the apparent maximum for each bounce. Record the coordinates of these points as (bounce number, bounce height). For example, if bounce 1 had a height of 6 units, the coordinates would be recorded as (1, 6).



1. When tracing the data, did you find the actual maximum of each bounce?

2. What is the asymptote for the ball bounce experiment?

3. Find a model for this new data set.

Teacher Notes



Activity 7

Modeling Exponential Decay with a Look at Asymptotes

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Materials

- TI-84 Plus/TI-83 Plus
- Sets of 6-sided dice (numbered 1–6)
- Alternate (non-standard) sets of dice

Time

- 90 minutes

In the first experiment, students use sample data to approximate models with the Transformation Graphing App. It provides a systematic look at modeling. Students should be encouraged to repeat this experiment in small groups and go through the same steps with their own data. This will allow for a rich discussion because each group will have different data sets.

In these experiments, students are introduced to the idea of discrete data sets being used with continuous function models. This concept is very important and should be a point of emphasis. Point out to students that they are using models to make sense of data. The data cannot be changed or forced to fit a model; rather they are to find a model to fit the data.

Stress with students the importance of accurate and reliable data. They should make sure that with every roll, all the dice truly roll.

Equipment for Experiments

For the first experiment, students will need sets of 36 regular dice (numbered 1–6). For experiment 2, students will need sets of 36 dice, some of which will not have a six. You can decide how many dice with and without sixes each group gets.

- You can get dice like these at joke, game, or magic stores. The best dice will have 1 through 5, with one number appearing twice.

- A second way to get dice like these is to make them with “blank” cubes that can be bought at party stores or mathematics supply houses. The cubes are called number cubes, and you put the numbers on yourself.
- Another way is to put colored dots on actual dice. Use six colors, and tell students to pick up a die when a blue dot appears on top. Be sure not to put any blue dots on some dice and one blue dot per die on most of them.
- You can also make them yourself from molding you can buy at a hardware supply store.

Be sure to have your students observe all the graphs of exponential decay so they can realize the difference between variation in the asymptote and in the value for B .

In both experiments, the students will most likely reach 15 rolls before all dice are removed. After they predict how many total rolls would be needed to remove all the dice, have them continue where they left off in the process. Remember that the second experiment will not go to zero. They roll until all dice with 6 have been removed.

The second activity may be done with or without the students being aware of the nature of the “special dice” set. Having students do the experiment “blind” provides an opportunity for a true investigation. You might give different groups different mixes from the “special dice” set. One group might have four of the non-six dice, and another may have two. This will allow for different non-zero asymptotes.

Answers

Modeling the Experiment: Casting Out Sixes

Data Analysis

4. The plot suggests an exponential decay model. Approximately $\frac{1}{6}$ of the dice should be removed each time, so there should to be a constant percent decrease. The curve will be pseudo-asymptotic to $y = 0$.
10. b. B is the growth or decay factor. B shows what percent of the dice should remain after each roll. Because there are six possible faces and only one has a six, the probability that a die will remain is $\frac{5}{6}$. Thus, $\frac{5}{6}$ of the dice should remain with each set of rolls. $\frac{5}{6} = 0.83$, which is the percentage that would be expected.

Modeling the Experiment: Casting Out Sixes with the Special Number Cubes**Data Analysis**

- The data show an exponential decay model. The curve of the plot is not as close to the x -axis in the second experiment so the data do not necessarily appear to be asymptotic to $y = 0$. Compare this plot with the data plotted in the last activity.
- The model seems reasonable when $A = 32$ and $C = 4$.
- The last die that had a six on it would be removed on approximately the twenty-third roll.

X	Y1
17	5.3473
18	5.1183
19	4.9282
20	4.7704
21	4.6394
22	4.5307
23	4.4405

X=23

A Method for Visualizing the Asymptote

- Some of the dice had no sixes, so they will never be removed from the sample data set.

Student Worksheet

- You do not necessarily have the maximum, but you have a value that is close to the maximum. Do not assume any special property for any point collected with a CBR™.
- The asymptote is 0. The ball will eventually stop bouncing. It is actually a pseudo-asymptote.
- Answers will vary, depending on the data set used; however, the model should be an exponential decay model.

