Serious Series

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Abstract: This activity is an introduction to infinite series. Students learn about geometric and telescoping series and the relationship between the sequence of partial sums of a series and its sum. They then use the symbolic capacity of their calculator and calculus to determine the sum of these series.

NCTM Principles and Standards:
Algebra standards
a) Understand patterns, relations, and functions
b) generalize patterns using explicitly defined and recursively defined functions;
c) use symbolic algebra to represent and explain mathematical relationships;
d) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.

Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard
a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

Key topic: Infinite series - geometric and telescoping series

Degree of Difficulty: Elementary
Needed Materials: TI-89 calculator
Situation: If we add the terms of a sequence together we get a series. The terms of a sequence are separated by commas, while the terms of a series can be separated by addition signs. Here we will be considering infinite series. The series $a_1 + a_2 + a_3 + a_4 + \ldots + a_n + \ldots$ can be written in summation form: $\sum_{n=1}^{\infty} a_n$.
If a series converges, we say it converges to its sum. Otherwise we say it diverges.
Some series are familiar:

$$\frac{36}{100}+\frac{36}{10000}+\frac{36}{1000000}+....=.363636...=.\overline{3}\overline{6}=\frac{4}{11}$$

To determine whether or not a series has a sum, we will form a special sequence from its terms. This sequence is called the **sequence of partial sums.** The first term of the sequence is the first term of the series. The second term of the sequence of partial

sums is the sum of the first two terms of the series. The third term of the sequence of partial sums is the sum of the first three terms of the series, etc.
So:
$s_1=a_1$
$s_2=a_1+a_2$
$s_3=a_1+a_2+a_3$
$s_4=a_1+a_2+a_3+a_4$
.
.
.
$s_n= a_1+a_2+a_3+a_4+\ldots+a_n=\sum_{i=1}^{n} a_i$

If the sequence of partial sums for a series converges to a limit S, then the series is said to converge to its sum S.

Here's an example:

Consider the series: $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\frac{1}{48}+\ldots$

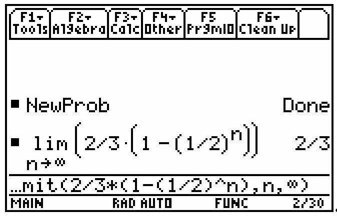
The first terms of the sequence of partial sums are:
$s_1=1/3 = .33333$
$s_2=1/3 + 1/6 = ½ = .50$
$s_3= 1/3 + 1/6 + 1/12= 7/12 =.58333$
$s_4=5/8 = .625$
$s_5=31/48 = .645833$
$s_6=21/32 = .65625$
$s_7=127/192 = .661458$

Clearly the sequence of partial sums appears to approach a limit of 2/3. Thus we say $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\frac{1}{48}+\ldots=2/3$

This particular series is geometric. Here's a brief review of geometric series. A geometric series has terms which have a common ratio, r. It can be expressed as:

$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$ where $a\neq0$. The formula for the sum of the first n terms of this geometric series is: $s_n=\frac{a-ar^n}{1-r}$. This is exactly the nth term of the sequence of partial sums for the geometric series. Therefore the general term of our series is $\frac{1}{3}*\left(\frac{1}{2}\right)^{n-1}$. The nth term of the sequence of partial sums is $\frac{2}{3}\left(1-\left(\frac{1}{2}\right)^{n}\right)$.

If the sequence of partial sums of a series is convergent and converges to a limit S, then we say that the series itself converges to a sum S. If the series doesn't converge, it is divergent.

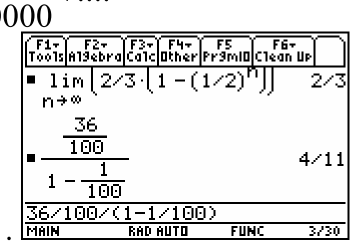
You can use your calculator to find the limit of the sequence of partial sums of this series in the following way: 

. The limit command is in the calculus menu and the ∞ symbol is: ◆ CATALOG

If |r|<1 where r is the ratio of a geometric series that series will converge to the sum: $\frac{a}{1-r}$ where a is the value of the first term.

$\frac{36}{100}+\frac{36}{10000}+\frac{36}{1000000}+....$ has a first term of 36/100 and a ratio of 1/100 so its sum is

$$\frac{\frac{36}{100}}{1-\frac{1}{100}}=\frac{36}{99}=\frac{4}{11}$$

. 

Another way of showing that a series is convergent is often referred to as telescoping. This method can be used only be used with some series, but nonetheless it illustrates an important mathematical technique. This refers to the way an old-fashion telescope collapses into a shorter length.

Consider the series:

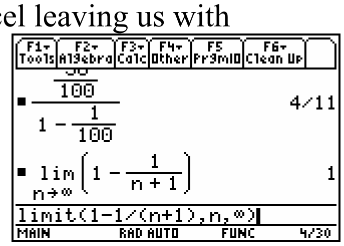
$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+...+\frac{1}{n(n+1)}+...$$

Now $\frac{1}{i(i+1)}=\frac{1}{i}-\frac{1}{i+1}$ *so we can write the* n^{th} *partial sum as*:

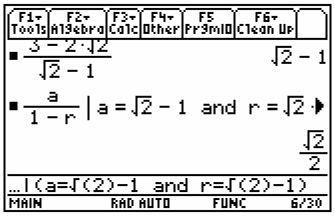
$$s_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+...+\left(\frac{1}{n}-\frac{1}{n-1}\right)$$

Notice that most of the fractions (in the middle) cancel leaving us with

$$s_n=1-\frac{1}{n+1},\ so\ \lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1-0=1$$

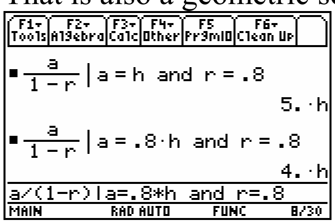


Example 1: Consider the geometric series: $\left(\sqrt{2}-1\right)+\left(3-2\sqrt{2}\right)+\ldots$ What is the common ratio and what is the sum of the series?



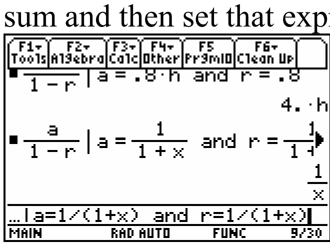
Example 2: A ball is dropped from a height h and bounces to 0.8 of its previous height after each bounce. What is the total distance traveled?

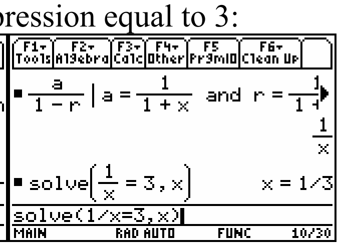
In this case we also have a geometric series whose first term is h and whose ratio is 0.8. We can find the total distance traveled as the ball goes down by finding the sum of this series. We also need to find the sum of the distances that the ball bounces up. That is also a geometric series whose first term is 0.8h whose ratio is also 0.8:



so the total distance traveled is 9h.

Example 3: What is the value of x if $\sum_{n=1}^{\infty}(1+x)^{-n}=3$? This is another geometric series where the first term and ratio are both $\frac{1}{1+x}$. We can find an expression for the sum and then set that expression equal to 3:



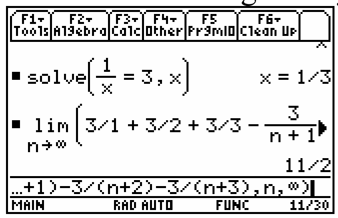


Example 4: Find the sum of $\sum_{1}^{\infty}\frac{9}{n(n+3)}$. This is another example of a telescoping series.

We can write out the first terms:

$$\frac{9}{1\cdot 4}+\frac{9}{2\cdot 5}+\frac{9}{3\cdot 6}+\frac{9}{4\cdot 7}+\ldots=\left(\frac{3}{1}-\frac{3}{4}\right)+\left(\frac{3}{2}-\frac{3}{5}\right)+\left(\frac{3}{3}-\frac{3}{6}\right)+\left(\frac{3}{4}-\frac{3}{7}\right)+\left(\frac{3}{5}-\frac{3}{8}\right)+\ldots$$

We can see that the sum of the first n terms can be given by the expression:



$$\frac{3}{1}+\frac{3}{2}+\frac{3}{3}-\frac{3}{n+1}-\frac{3}{n+2}-\frac{3}{n+3}$$

Problems:

1) What is the value of x if $\sum_{n=1}^{\infty}(1+2x)^{-n} = 12$?

2) Let S = 3 + 6(2-x) + 12(2-x)^2 + 24(2-x)^3+...

a) For which values of x does S have a sum?

b) Find the value of x such that S = 6

3) Let S = 3 + 12x + 48x^2 + 192x^3 + ... Find a value for x for which S = 4

4) Find r for the infinite geometric series which has a sum of

$$\frac{4+3\sqrt{2}}{2} \; where \; a = \sqrt{2}+1$$