Concepts

The purpose of this document is to introduce the accumulation, or area-so-far, function, and to help discover graphically the connection between differential and integral calculus. The first part of the Fundamental Theorem of Calculus involves a function defined by

$$g(x) = \int_a^x f(t)dt$$

where f is a continuous function on $[a,b]$ and x varies between a and b. The function g depends only on the value of x, and as x varies $g(x)$ also varies. Therefore, g is a function of x.

On Page 1.2, the user can manipulate the values of x and a. The value of $g(x)$ is dynamically computed and displayed, and the geometric interpretation of accumulated area, or area-so-far, is also shown as a shaded region on the graph. There are five points on the graph that can be moved vertically to change the definition of the piecewise defined linear function f.

Page 2.2 presents a similar graph and calculations: the graph of a piecewise defined linear function f, and the value $g(x)$ in the left pane. The bottom pane displays a complete graph of g. The values of x and a can be changed by grabbing and moving along the horizontal axis, or by using the sliders in the left pane. The dynamic connection between the two graphs and the vertical alignment allows the user to discover the relationship between the functions g and f.

Course and Exam Description Unit

Section 6.4: The Fundamental Theorem of Calculus and Definite Integrals

Calculator Files

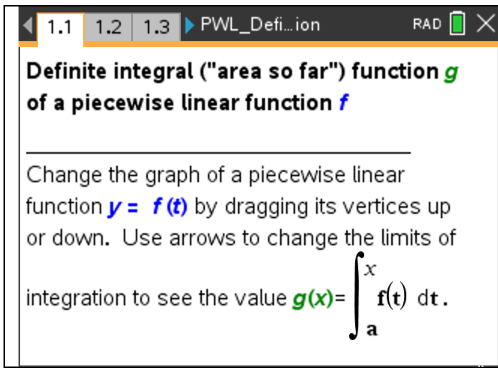
PWL_Definite_Integral_Function.tns

Using the Document

PWL_Definite_Integral_Function.tns: On page 1.2, a function f is presented as a piecewise defined linear graph. The vertices that connect the linear pieces of the graph can be moved up or down (in integer steps) by grabbing any marked point on the graph and dragging to another location. The values of x and a can also be changed by grabbing the corresponding point and dragging along the horizontal axis. These values can also be manipulated by using the sliders in the left pane. For a fixed value a, the value of $g(x)$ is displayed in the bottom pane.

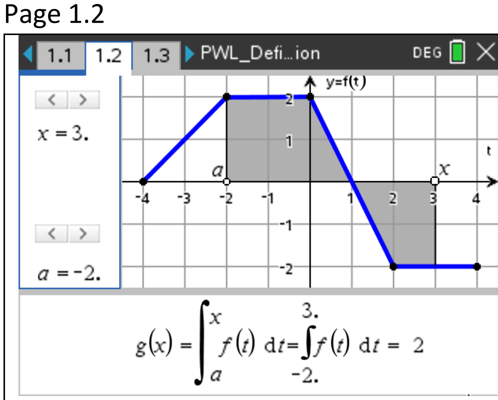
On page 2.2, a function f is presented as a piecewise defined linear graph in the top pane. The vertices can be moved up or down, and the values of x and a can be changed on the graph or by using the sliders. The graph of the function g is displayed in the bottom pane.

Page 1.1



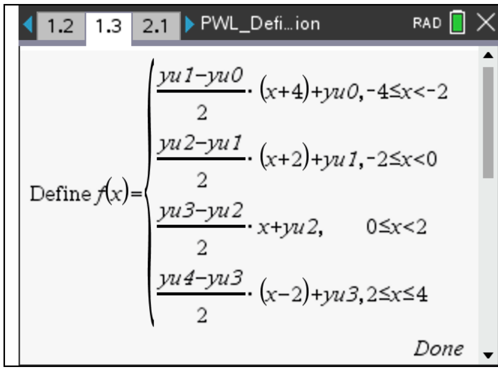
In Problem 1, the integral of a piecewise defined linear function f is considered. The vertices that connect the linear pieces of the graph of f can be moved vertically, in integer steps, by grabbing any marked point on the graph and dragging to another location. The slider arrows can be used to change the limits of integration, a and x. The user can also grab and move the points on the graph representing a and x.

Page 1.2



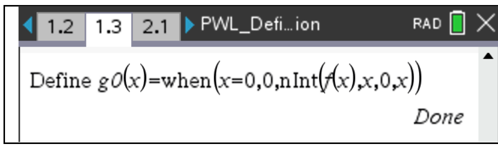
The graph of a function f for $-4 \le t \le 4$ is shown in the top right pane. The vertices that connect the linear pieces of the graph can be moved up or down (in integer steps) by grabbing the point and dragging to another location. The slider arrows in the left pane can be used to change the values of a and x in steps of 0.1. The user can also grab and move the points on the graph representing a and x in the top pane. The value of $g(x)$, for the current values of a and x, is given in the bottom pane. This value updates dynamically, that is, as a or x changes. The shaded region in the graph is a geometric interpretation of the value of $g(x)$.

Page 1.3 (top)



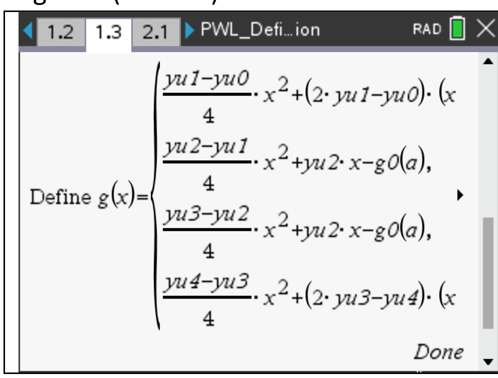
This page contains the definitions for the functions f and g. The calculator variables $yu0, yu1, \ldots, yu4$ represent the y-coordinates of the points on the graph of f. The user does not need to alter these definitions. However, one might consider deriving this definition for f.

Page 1.3 (middle)



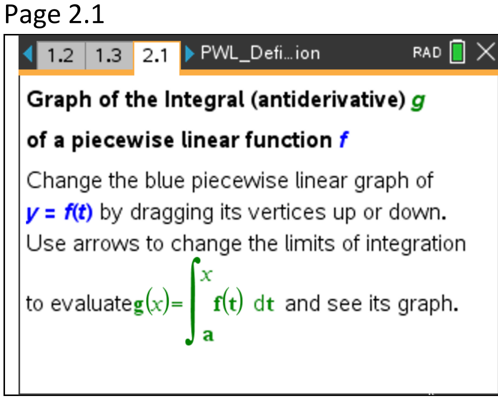
In order to speed up the calculations (and the display in Problem 2), the function g is defined analytically. The function $g0$ is used in the definition of g.

Page 1.3 (bottom)



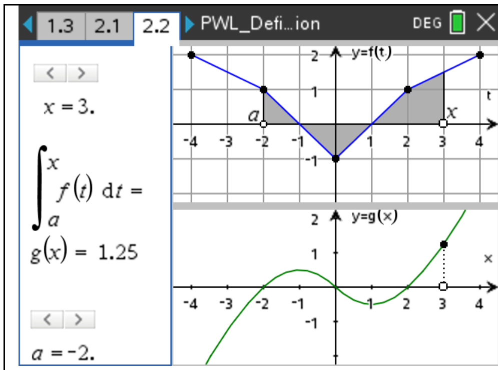
This is the analytical definition for the function g. The user does not need to change any of these functions. However, they are provided in case the user would like to add more nodes or enhance this file in some other way.

Page 2.1



In Problem 2, the integral of a piecewise defined linear function f is considered, and the graph of the function g is also displayed. The vertices that connect the linear pieces of the graph of f can be moved vertically by grabbing any marked point on the graph and dragging to another location. The slider arrows can be used to change the limits of integration, a and x. The user can also grab and move the points on the graph representing a and x. The graph of the function g is displayed dynamically.

Page 2.2



The graph of a function f for $-4 \le t \le 4$ is shown in the top right pane. The vertices that connect the linear pieces of the graph can be moved up or down (in integer steps) by grabbing the point and dragging to another location. The slider arrows in the left pane can be used to change the values of a and x in steps of 0.1. The user can also grab and move the points on the graph representing a and x in the top pane. In the bottom pane, the user can grab and move the point representing x. The value of $g(x)$, for the current values of a and x, is given in the left pane. This value updates dynamically, that is, as a or x changes. The shaded region in the graph is a geometric interpretation of the value of $g(x)$. The graph of g is displayed in the bottom right pane. This graph changes dynamically as a or x change.

Page 2.3

This page contains definitions for the functions f and g, similar to those on Page 1.3. The user does not need to alter any of these definitions. This page is provided for consideration and added discussion, and in case the user would like to enhance this calculator file in some way.

Suggested Applications and Extensions

Page 1.2

Use the default function f to answer Questions 1-8. Remember that g is a function of x (for a fixed value of a). The values of a and x can be manipulated, the value $g(x)$ is displayed in the bottom pane, and the shaded region in the top pane represents the accumulated net area bounded by the graph of f and the horizontal axis from a to x.

1. Find the domain and range of the function g.
2. Use the graph to explain geometrically how to find $g(0)$, $g(1)$, $g(2)$, and $g(3)$.
3. On what intervals is g increasing? Decreasing?
4. Find $g(-4)$. Explain this answer since the shaded region representing $g(-4)$ is above the horizontal axis.
5. On the interval $[-4,4]$ where does g have an absolute maximum value? Explain the behavior of the function f at and around this value.
6. On the interval $[-4,4]$ where does g have an absolute mimimum value? Does this contradict the Extreme Value Theorem? Why or why not?
7. Let $a = -4$. Explain how the values in Question 1 change.

8. Let $a = 0$. Explain how the values in Question 1 change.
9. Let $a = -4$. Move the points to construct a piecewise defined linear function such that the maximum value of g occurs at $x = 0$. Let $a = -2$. Where does the absolute maximum occur now?
10. Let $a = 0$. Move the points to construct a non-zero piecewise defined linear function such that $g(4) = 0$. Let $x = 4$. Explain the relationship among the shaded regions in the graph of f to the left of $x = 0$.
11. Let $a = 0$. Move the points to construct a non-zero piecewise defined linear function such that the function g is increasing on the interval $[-4, 4]$. In words, describe any special characteristic of your function f. Does this suggest another relationship between g and f? If so, explain.
12. Let $a = -4$. Is it possible to construct a non-zero piecewise defined linear function such that $g(4) = 25$? If not, why not? If so, construct one such function.

Page 2.2

Use the default function f to answer Questions 1-9. Remember that g is a function of x (for a fixed value of a). The values of a and x can be manipulated, the value $g(x)$ is displayed in the left pane, and the shaded region in the top pane represents the accumulated net area bounded by the graph of f and the horizontal axis from a to x. The bottom pane displays a complete graph of the function g.

1. Find the domain and range of g.
2. Find the value $g(-1)$. Explain how to determine this value geometrically.
3. Find the value $g(0)$. Explain this answer geometrically.
4. On what intervals is g increasing? Decreasing? What are the values of f on each of these intervals?
5. For what values of x does g have a relative minimum value? Relative maximum value?
6. Where does g attain its absolute maximum value? Absolute minimum value?
7. On what intervals is g concave up? Concave down? Explain the behavior of the function f on each of these intervals.
8. Find any points of inflection on the graph of g. Explain the behavior of the graph of f at each corresponding x-coordinate.
9. Use your answers to questions 1-8 to suggest a relationship between g and f.
10. Let $a = 0$. Move the points to construct a non-zero piecewise defined linear function such that the function g is increasing on the interval $[-4, 4]$. In words, describe any special characteristic of your function f.
11. Let $a = 0$. Move the points to construct a non-zero piecewise defined linear function such that the graph of the function g has two relative maximum points and two relative minimum points.

12. Let $a = 0$. Is it possible to construct a non-zero piecewise defined function such that the graph of the function g has three relative maximum points? If not, why not? If so, then construct one such graph.
13. Let $a = -2$. Move the points to construct a non-zero piecewise defined linear function such that the graph of the function g is concave down over its entire domain.
14. For your graph constructed in Question 13, explain what happens to the graph of g as a changes.
15. Let $a = 0$. Move the points to construct a non-zero piecewise defined linear even function. Is the function g even, odd, or neither?
16. Let $a = 0$. Move the points to construct a non-zero piecewise defined linear odd function. Is the function g even, odd, or neither?