

Teacher Notes



Activity 6

Local Linearity, Differentiability, and Limits of Difference Quotients

Objectives

- Connect the concept of local linearity to differentiability through numerical explorations of limits of difference quotients
- Develop a flexible, coordinated, multi-representational image of the limiting process as it relates to average rates of change passing to instantaneous rates of change

Materials

- TI-84 Plus / TI-83 Plus

Teaching Time

- 100 minutes

Abstract

In this activity, the local linearity of several functions at different points is discussed and explored. The concept of local linearity is linked to differentiability by examining one-sided limits of difference quotients of several functions numerically as h approaches zero. By the end of the activity, students will see examples of finite, one-sided difference quotient limits that are not equal, and one-sided difference quotient limits that approach infinity. These behaviors (as well as difference quotient limits that exist) are linked to the graphical behavior of the function. This activity contains 20 questions and 2 extension exercises. Most questions require students to interpret the information displayed on the graphing handheld using the result of a graphical or numerical exploration. A few questions may require some algebraic manipulation.

Management Tips and Hints

Prerequisites

- Students should be able to graph and modify (zoom, trace, and so on) graphs of functions and generate tables using the graphing handheld.
- This activity is appropriate after the derivative and differentiability are defined.

Evidence of Learning

Students can investigate the local linearity given a function and a point and then connect that notion with the function's differentiability at that point.

Common Student Errors/Misconceptions

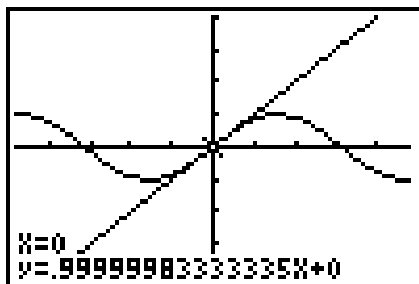
- The difference quotient uses two literal symbols, x and h . Moreover, the value of x is often fixed at some arbitrary real number often symbolized as a . Students often confuse the roles of these symbols.
- It is necessary to use T in place of h when exploring on the graphing handheld. Questioning students about the roles and values of x and h as they work through the activity may help students examine the roles more closely.

Extensions

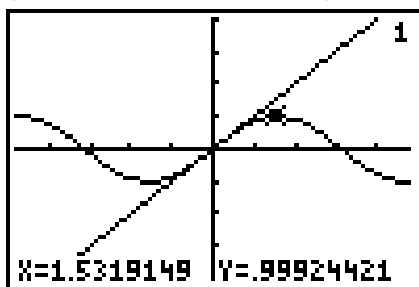
1. Driver Dan claims that during a one-hour trip his displacement function is $s(t) = |t - 30|$ where t is measured in minutes and s is measured in miles. Is this possible? Explain.
2. For Driver Dan's alleged trip:
 - a. Draw the displacement versus time graph.
 - b. Draw the velocity versus time graph.

Activity Solutions

1. No; it does not show the sine function's wider properties. The window is so small that the graph looks like a line.
2. $y = 0.99999983333335x + 0$, which is very close to $y = x$.
3. Yes



4. (1.5319149, 0.99924421)



5. Vertical distance at $x = 1.5319149$
 - To line: $(0.99999983333335)(1.5319149) = 1.531914645$
 - To curve: $\sin(1.5319149) = 0.9992442125$
 - Difference: 0.5326706875
6. Points will vary, but $f(x) = \sin(x)$ should appear locally linear everywhere.
7. No; no amount of zooming in will get rid of the sharp corner.
8. No; there is a cusp at (1, 0).
9. Yes; the difference quotient appears to approach 1 as T approaches 0 from the right. Students should look from the top of the table down toward $T = 0$.

T	X_{1T}	Y_{1T}
.3	0	.98507
.2	0	.99335
.1	0	.99833
0	0	ERROR
-.1	0	.99833
-.2	0	.99335
-.3	0	.98507

T=.3

10. Yes; the difference quotient appears to approach 1 as T approaches 0 from the left. Students should look from the bottom of the table up toward $T = 0$.

11. Yes

12. The difference quotient is getting larger. It approaches infinity as T approaches 0 from the right.

T	X_{1+T}	Y_{1+T}
.03	1	3.2183
.02	1	3.684
.01	1	4.6416
0	1	ERROR
-.01	1	-4.642
-.02	1	-3.684
-.03	1	-3.218
T = .03		

13. The difference quotient is getting smaller. It approaches negative infinity as T approaches 0 from the left.

14. The limit does not exist because both one-sided limits are not finite and not equal to each other.

This means that $f(x) = \sqrt[3]{(x-1)^2}$ is not differentiable at $x = 1$ or that f is not locally linear at $(1, 0)$.

15. The difference quotient approaches 2 as T approaches 0 from the right.

T	X_{1+T}	Y_{1+T}
.03	1	2.03
.02	1	2.02
.01	1	2.01
0	1	ERROR
-.01	1	1.99
-.02	1	1.98
-.03	1	1.97
T = .03		

16. The difference quotient approaches 2 as T approaches 0 from the left.

17. The one-sided limits are equal to each other, so the limit may exist.

18. Yes; the difference quotient is 1 for all positive T .

T	X_{1+T}	Y_{1+T}
.03	2	1
.02	2	1
.01	2	1
0	2	ERROR
-.01	2	-1
-.02	2	-1
-.03	2	-1
T = .03		

19. Yes; the difference quotient is -1 for all negative T .

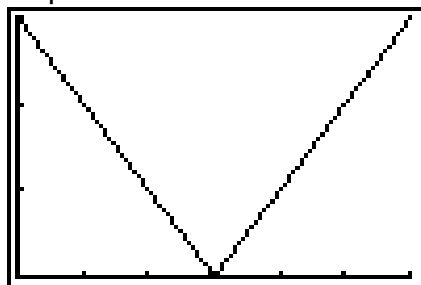
20. No; the one-sided limits are not equal to each other.

Extension Solutions

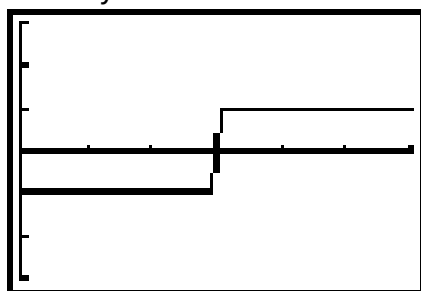
1. Dan's trip is impossible because his displacement function would dictate an instantaneous velocity change from -1 miles per minute to $+1$ miles per minute.

- 2.

- a. Displacement versus Time



- b. Velocity versus Time



Activity 6

Local Linearity, Differentiability, and Limits of Difference Quotients

Introduction

In this activity, you will explore a graphical feature of some functions called local linearity. You will then investigate the link between local linearity and differentiability through several examples.

Exploration

Begin by generating the graph of $f(x) = \sin(x)$ in a very specific viewing window. First, make sure that your graphing handheld is in **Radian** mode.

Set the viewing window as shown.

```

WINDOW
Xmin=-.3
Xmax=.3
Xscl=1
Ymin=-.25
Ymax=.25
Yscl=1
Xres=1
  
```

Input **Y1** as **sin(X)**, and **GRAPH** the function.

1. Does this generate a good drawing of a sine curve? Why does this graph look the way it does?

Now, draw the tangent line to the graph of $f(x) = \sin(x)$ at the point $(0, 0)$ in this same window. Press **TRACE**, select **5:Tangent(** from the **DRAW Menu**, and then press **ENTER**.

Objectives

- Connect the concept of local linearity to differentiability through numerical explorations of limits of difference quotients
- Develop a flexible, coordinated, multi-representational image of the limiting process as it relates to average rates of change passing to instantaneous rates of change

Materials

- TI-84 Plus / TI-83 Plus

2. Write the equation of the line that appears at the bottom of the screen.

Check to make sure that the zoom factors are both set to 4 (select **4: SetFactors...** in the **ZOOM MEMORY Menu**), select **3:Zoom Out** in the **ZOOM Menu**, and then press $\boxed{\text{ENTER}}$.

You should now be able to see a small curve in the graph, but the tangent line disappeared. Because **DRAW** commands appear on the graph only temporarily, after the window is changed, the tangent line does not re-**DRAW**. You could type the equation of the tangent line into **Y2**, but for now just select **3:Zoom Out** in the **ZOOM Menu** one more time. Press $\boxed{\text{ENTER}}$, select **5:Tangent(** from the **DRAW Menu**, and then press $\boxed{\text{ENTER}}$ again.

3. Did the graphing handheld return the same equation for the line as in Question 2?

You can say that a function f is *locally linear* at a point $(a, f(a))$ if it looks like a line the more you zoom in on it at the point $(a, f(a))$. In the previous example, $f(x) = \sin(x)$ appears to be locally linear at the point $(0, 0)$. In other words, in a small x -interval around $x = 0$, the graph of $f(x) = \sin(x)$ looks just like the graph of the line tangent to $f(x) = \sin(x)$ at $(0, 0)$.

Of course, away from the point $(0, 0)$, the graph of $f(x) = \sin(x)$ begins to stray from the line tangent to f at $(0, 0)$. Given an input value other than 0, for instance $x = a$, the difference between the value of $f(a)$ and the y -value that corresponds to $x = a$ on the line tangent to f at $x = 0$ must be analyzed. If the line tangent to f at $x = 0$ is used to approximate the function at $x = a$, it is necessary to determine how much error would result.

Trace along the graph of $f(x) = \sin(x)$ until you reach the point nearest the first maximum (near $x = \frac{\pi}{2}$). Use the x -value of this point as $x = a$.

4. What are the coordinates of the point where the crosshairs are located?
5. What is the vertical distance between the graph of $f(x) = \sin(x)$ and the tangent line at the point in Question 4? (Hint: Look at the equation in your answer to Question 2.)

To determine whether $f(x) = \sin(x)$ is locally linear at other points, investigate the local linearity at one other point.

Trace to some other point on $f(x) = \sin(x)$. Select **2:Zoom In** from the **ZOOM Menu**, and press $\boxed{\text{ENTER}}$. Repeat this process a few times until you are convinced that other local linearity points exist. Be sure to press $\boxed{\text{TRACE}}$ before each zoom to make sure that the screen will be centered on the point you have chosen.

6. At what point did you test the local linearity of $f(x) = \sin(x)$? Does it appear to be locally linear there?

Some functions have points that are not locally linear.

Let $f(x) = |x - 1| + 1$. Generate a graph of this function in the **Zdecimal** viewing window.

TRACE $f(x) = |x - 1| + 1$ to the point $(1, 1)$, select **2:Zoom In** from the **ZOOM Menu**, and press **ENTER**. Repeat this process a few times until you are convinced that f is or is not locally linear at $x = 1$. Be sure to press **TRACE** before each zoom to make sure that the screen will be centered on $(1, 1)$.

7. Does $f(x) = |x - 1| + 1$ appear to be locally linear at $(1, 1)$? Explain your answer.
8. Try the same graphical approach with:

$$f(x) = \sqrt[3]{(x-1)^2} = (x-1)^{(2/3)}$$

Is this function locally linear at $(1, 0)$? Explain your answer. (Hint: You may want to turn the axes off for this question.)

“Tracing and zooming are good ways to graphically investigate local linearity. The last two exercises should also suggest that a function is not locally linear at any point where the derivative of the function does not exist. In other words:”

A function f is locally linear at the point $(a, f(a))$ if and only if the following limit exists:

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

To understand why this relationship exists, it is helpful to get a numerical perspective of how this difference quotient behaves as h approaches zero. To investigate the difference quotient for $f(x) = \sin(x)$ at $(0, 0)$ numerically, you can begin by building a table containing the difference quotient for values of h close to (and on both sides of) zero. In this situation, it is convenient to use parametric equations to represent the difference quotient and the point on the function. On the graphing handheld, the parameter T is used in place of h .

- Set the graphing handheld into **Par** mode.
- Enter the x -coordinate of the point $(a, f(a))$ and the right difference quotient into the first set of parametric equations. Enter the equations below in the **Y=** editor:

$$X_{1T} = 0$$

$$Y_{1T} = \frac{\sin(X_{1T} + T) - \sin(X_{1T})}{T}$$

Keep in mind that you need to get the special variable X_{1T} by selecting **1: X_{1T}** from the **2:Parametric...** secondary menu in the **VARS, Y-VARS Menu**. The finished screen should look like the screen shown.

```

Plot1 Plot2 Plot3
\X1T=0
Y1T=(sin(X1T+T)
-sin(X1T))/T
\X2T=
Y2T=
\X3T=
Y3T=

```

- Set up the table as shown.

```

TABLE SETUP
TblStart=.3
ΔTbl=-.1
Indent: Auto Ask
Depend: Auto Ask

```

- View the table.
9. Do the values of the difference quotient Y_{1T} approach any particular number as T approaches 0 from the right? If so, what is it? Are you looking from the top of the table down or from the bottom of the table up when T approaches 0 from the right?
 10. Do the values of the difference quotient Y_{1T} approach any particular number as T approaches 0 from the left? If so, what is it? Are you looking from the top of the table down or from the bottom of the table up when T approaches 0 from the left?

Change the table settings by a factor of 10 to look at values of T even closer to zero. Set the table as shown.

```

TABLE SETUP
TblStart=.03
ΔTbl=-.01
Indent: Auto Ask
Depend: Auto Ask

```

11. Do you still agree with your answers to Questions 9 and 10?

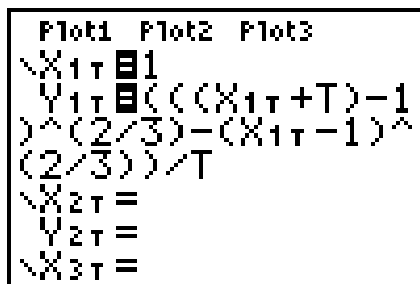
Examine a function at a point where the function is not locally linear.

Let $f(x) = \sqrt[3]{(x-1)^2}$ and choose the point (1, 0). Input the equations below into the Y= editor:

$$X_{1T} = 1$$

$$Y_{1T} = \frac{((X_{1T} + T) - 1)^{2/3} - (X_{1T} - 1)^{2/3}}{T}$$

The screen should look like the one shown.



View the table.

- 12.** Do the values of the difference quotient Y_{1T} approach any particular number as T approaches 0 from the right?
- 13.** Do the values of the difference quotient Y_{1T} approach any particular number as T approaches 0 from the left?
- 14.** Does the following exist when $a = 1$? Why or why not?

$$\lim_{h \rightarrow 0} \frac{\sqrt[3]{((a+h)-1)^2} - \sqrt[3]{(a-1)^2}}{h}$$

- 15.** Follow the same procedures with $f(x) = x^2$ and the point (1, 1).

Do the values of the difference quotient Y_{1T} approach any particular number as T approaches 0 from the right?

- 16.** Do the values of the difference quotient Y_{1T} approach any particular number as T approaches 0 from the left?

- 17.** Does the following exist when $a = 1$? Why or why not?

$$\lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h}$$

18. Follow the same procedures with $f(x) = |x - 2|$ and the point $(2, 0)$.

Do the values of the difference quotient Y_{1T} approach any particular number as T approaches 0 from the right?

19. Do the values of the difference quotient Y_{1T} approach any particular number as T approaches 0 from the left?

20. Does the following exist when $a = 2$? Why or why not?

$$\lim_{h \rightarrow 0} \frac{|(a + h) - 2| - |a - 2|}{h}$$