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Slope/Triangle Area Exploration

ID:

Time required
60 minutes

Topics: Linear Functions, Triangle Area, Rational Functions

- *Graph lines in slope-intercept form*
- *Find the coordinate of the x- and y-intercepts of a line.*
- *Calculate the area of a triangle.*
- *Identify vertical, horizontal, and oblique asymptotes for a rational function.*
- *Find values of functions on a graph.*

Activity Overview

In this activity, students investigate the following problem (Adapted from Philips Exeter Academy, 2007):

The equation $y - 5 = m(x - 2)$ represents a line, no matter what value m has.
(a) What are the x- and y-intercepts of this line?
(b) For what value of m does this line form a triangle of area 36 square units with the positive x- and y-axes?
(c) Describe the areas of the first-quadrant triangles.

Using multiple representations afforded by the TI-Nspire – in particular, dynamic geometry, spreadsheets, scatter plots, and CAS – students will explore how the area of a triangle formed by the x- and y-axes and a line through the point (2,5) is related to the slope of the line.

This activity is appropriate for students in Algebra 1, Geometry, and Algebra 2. It is assumed that students understand how to graph lines in point-slope form and that students know how to find the area of a triangle. For students in Algebra 2, knowledge of rational functions and asymptotes is helpful. However, knowledge of rational functions is not a prerequisite for this activity. In fact, this activity can be used as an introduction to rational functions by providing concrete examples of vertical asymptotes, oblique asymptotes, and zeros. This activity will allow students to begin to explore the concept of a limit and of rates of change in an informal manner.

Questions for further investigation will allow students to explore similar triangles, triangle centers, and to formulate generalizations relating the coordinates of the pivot point to the minimum triangle area.

Teacher Preparation

- *The screenshots on the following pages provide a preview of the TI-Nspire document, as well as expected student results for each problem.*
- *If you are planning on having your students investigate this activity individually or in pairs, you will need to download the .tns files to the student handhelds either before hand or at the start of class using Connect-to-Class.*
- *You may want to copy the guided and inquiry questions from this word document into another word document so you can distribute them to your students.*

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Classroom Management

- *Pages in this .tns activity can be used students individually or in pairs. The entire activity is appropriate for a whole-class format using the computer software and the questions found in this document. Depending on your students, the only part of this activity that may require teacher guidance is when students are deriving the triangle area formula.*
- *All the guided inquiry questions related to this investigation are on notes pages in the .tns file and are written in this activity guide. You may wish to have your students write their answers to these problems on a separate piece. You may find it best to copy these questions from this activity guide into another word document and distribute the questions to your students.*
- *The guided inquiry questions can be used to engage in a whole-class discussion. In addition to possible content specific things that may be addressed in this activity, there are informal introductions to pre-calculus and Calculus concepts that should be pointed out..*
- *Optional investigations are provided at the end of this activity in Problems 4 and 5 of the student .tns file. you may delete these problems from the .tns file.*

TI-Nspire™ Applications

Graphs & Geometry, Lists & Spreadsheet, Notes, CAS Calculator.

Reference

Philip Exeter Academy (July 2007). *Mathematics 2 (Problem 1, page 25).* Available on-line at http://www.exeter.edu/documents/math2all.pdf

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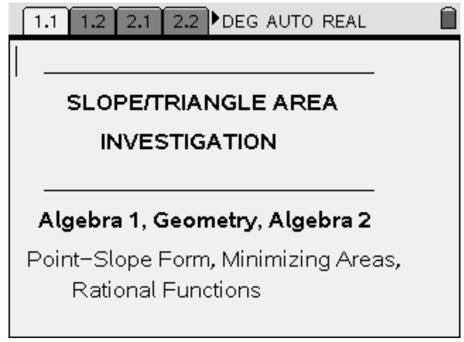
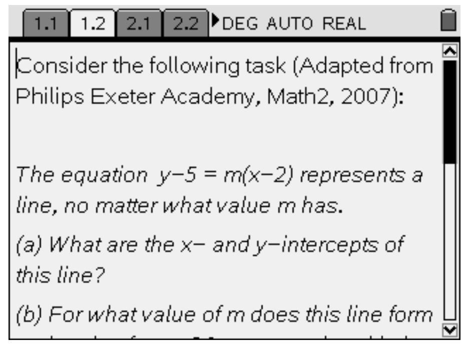
Problem 1 – Problem Introduction

The first two pages of this document introduce the problem. Students should be encouraged to explore this problem *by hand* prior to working through the document. One suggestion is a whole-class activity in which pairs of students calculate the intercepts and areas for random values of the slope of the line. This data is then organized in a table for the entire class to analyze. Guided inquiry questions are:

1. As the slope changes, how does the *y*-intercept change?
2. As the slope changes, how does the *x*-intercept change?
3. As the slope changes, how does the triangle area change?

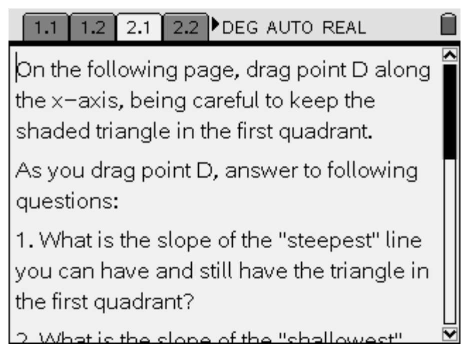


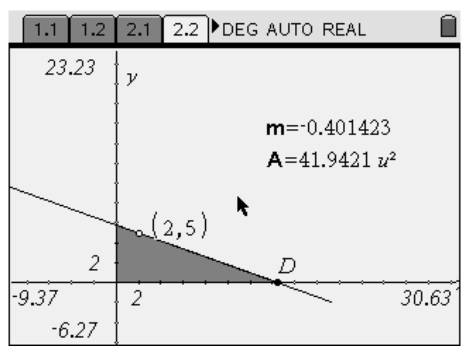


Problem 2 – Initial Investigation

The next two pages (2.1 and 2.2) allow students to explore a dynamic first-quadrant triangle. The questions on page 2.1 are intended to get the students to focus on manipulating the slope (ultimately, the independent variable in this problem) and its effect on the slope.

1. What is the slope of the "steepest" line you can have and still have the triangle in the first quadrant?
2. What is the slope of the "shallowest" line you can have and still have the triangle in the first quadrant?
3. Where is the triangle when the slope is positive?
4. What happens to the triangle when the slope is zero?
5. What happens to the triangle when the slope is undefined?
6. What values of the slope always produce a triangle in the first quadrant?
7. What value(s) of the slope produces a triangle with the smallest possible area?
8. What value(s) of the slope produces a triangle with an area of 36 square units?





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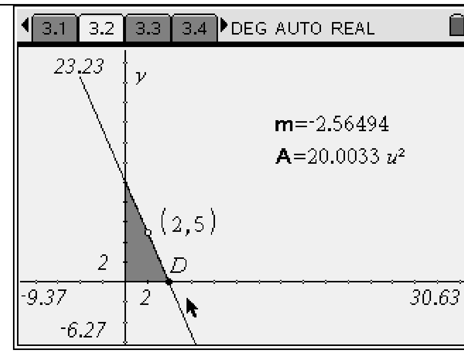
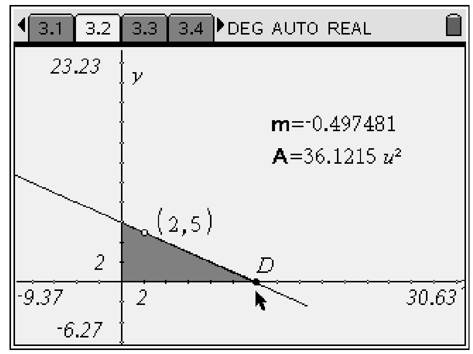
Page 3.2 at right illustrates an anticipated student response to question #7 displaying an area close to 20 square units when the slope is close to −2.5.

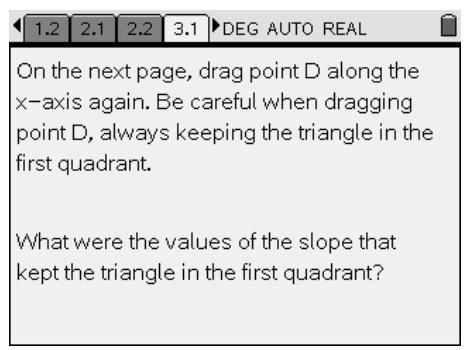


Page 3.2 at right illustrates an anticipated student response to question #8 displaying an area close to 36 square units when the slope is close to −0.5.

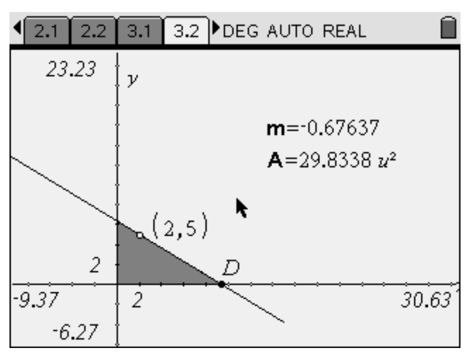


Problem 3 – Multiple Representations

These pages – there are 17 of them – are intended to lead to students through an exploration of this problem from multiple points of view.



Page 3.2 is the page that generates the slope and area data for the entire problem. As students drag point D to generate the data, they should drag point D *slowly*, being sure to generating data for just the first-quadrant triangle.



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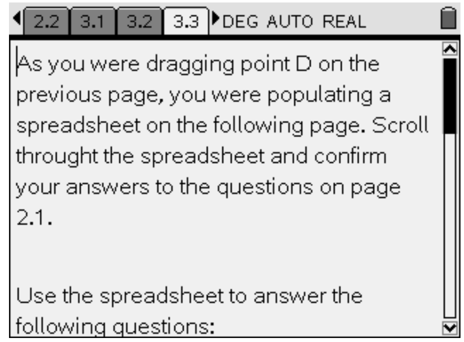
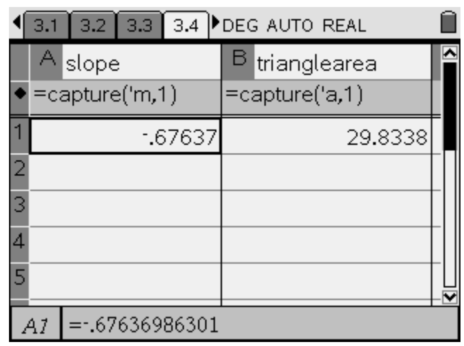
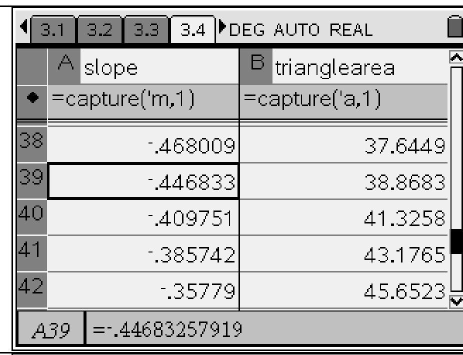
Pages 3.3 and 3.4 – Exploring data in a spreadsheet.
The data generated on page 3.2 is displayed here. Guided questions for exploring this data are:

1. Find the slope of a line that produces a triangle with an area close to 40 square units.
2. Find the slope of a line that produces an area close to 30 square units.
3. Find a slope that is close to -2 and another that is close to -3. What is the difference in the areas of the triangles?
4. Find a slope that is close to -4 and another that is close to -5. What is the difference in the areas of the triangles?
5. Find a slope that is close to -0.2and another that is close to -0.3. What is the difference in the areas of the triangles?
6. Find a slope that is close to -0.4 and another that is close to -0.5. What is the difference in the areas of the triangles?
7. For what values of the slope does the area of the triangle appear to be changing the fastest?

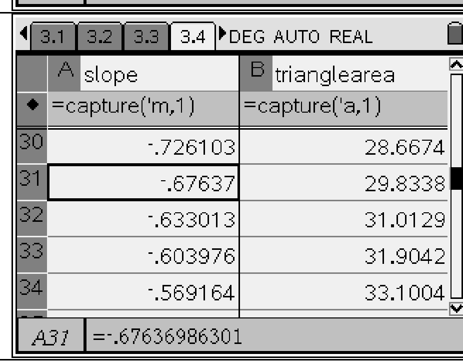




Page 3.4 at right shows an anticipated student response to question #1, displaying an area of 38.8683 square units when the slope is −0.446833.



Page 3.4 at right shows an anticipated student response to question #2, displaying an area of 31.0129 square units when the slope is −0.67637.



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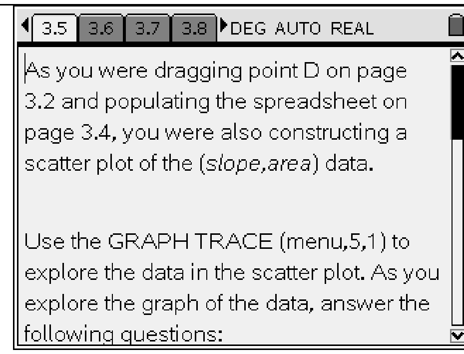
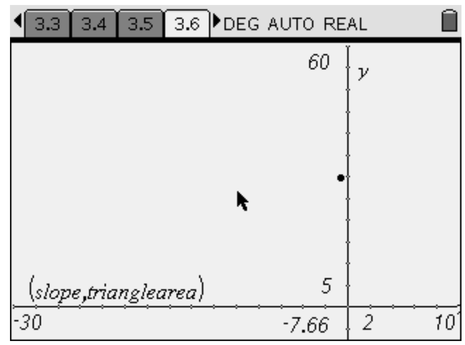
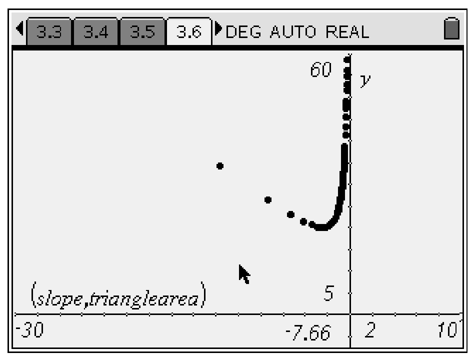
Paged 3.5 and 3.6 – Exploring data in a scatter plot
On these pages, the data generated on page 3.2 and collected in the spreadsheet on page 3.4 is displayed in a scatter plot on page 3.6. Questions for guided inquiry are:

1. The data looks like it gets very close to the y-axis. Will the graph of the data ever cross the y-axis? Use the slope of the line and the area of the triangle to explain your answer.
2. Will the graph of the data ever cross the negative x-axis? Use the slope of the line and the area of the triangle to explain your answer.
3. Why does the graph of the data show that there are two triangles with an area of 22 square units?
4. The data appears to have a "U" shape. Explain why the slope of the line and the area of the triangle lead to data that has a "U" shape.
5. <u>THOUGHT EXPERIMENT</u>: What do you think would happen to the graph of the data if you used the point (5,2) as the pivot point instead of the point (2,5)? Explain your reasoning using the slope of the line and the area of the triangle.



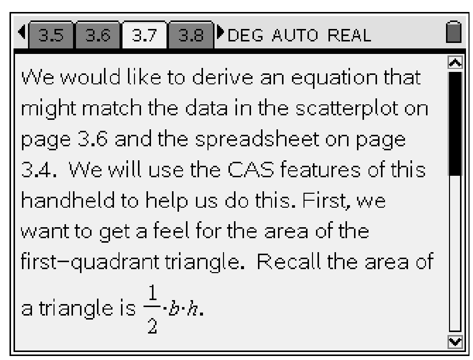


Page 3.6 at right shows an anticipate student scatter plot.



Pages 3.7 through 3.10 – Deriving an area formula using CAS
On the next four pages, students are guided through a derivation of a formula for the area of the triangle in terms of the slope of the line. This formula is stored at *f1*(*x*) and is used to answer questions in the problem. Guided questions prior to the derivation are intended to have students consider what goes into calculating the area of a triangle.

1. Using the axis intercepts of the line, what is the formula for the area of the first quadrant triangle?
2. In terms of the axis intercepts of the line, when



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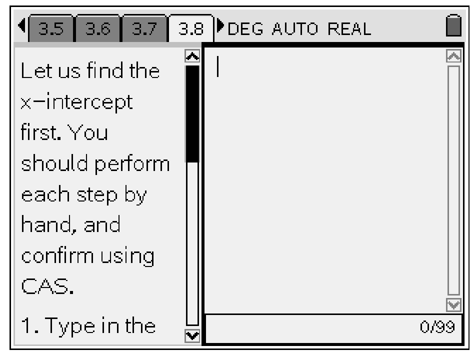
would two triangles have the equal areas?

3. If a triangle had an area of 36 square units, what are some possible values of the x- and y-intercepts?
4. Which of these intercept values are possible for THIS problem?

When students solve for the intercepts on pages 3.8 and 3.9, they should be reminded that they are solving for *x* (or *y*) in terms of the slope of the line. Students should do as much solving as possible *by hand* and confirm or check their results with CAS.

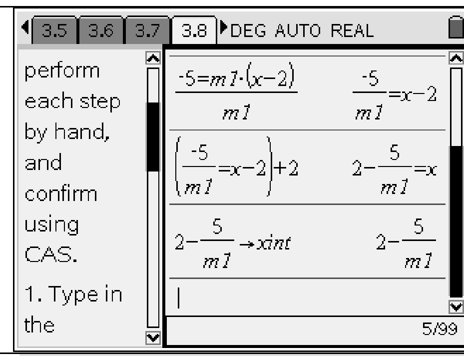
Page 3.8 – Solving for the *x*-intercept.

1. Type in the equation of the line. Use *m1* for the slope.
2. Substitute *y*=0 and solve for *x* in terms of *m1*.
3. There was a small error message that appeared. Why?
4. Store what x is equal to at *xint*.



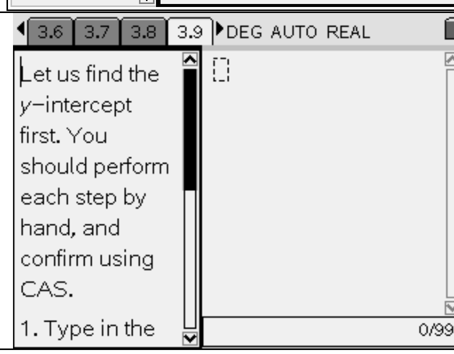
Page 3.8 at right shows an anticipated derivation of the x-intercept in terms of m1.

When you divide by m1, you will see an error message regarding the size of the domain. This is something that should be discussed with your students.



Page 3.9 – Solving for the *y*-intercept.

1. Type in the equation of the line. Use *m1* for the slope.
2. Substitute *x*=0 and solve for *y* in terms of *m1*.
3. Store what *y* is equal to at *yint*.



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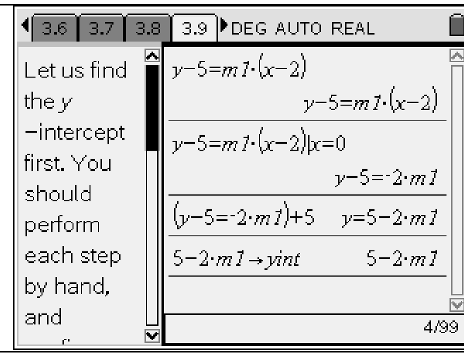
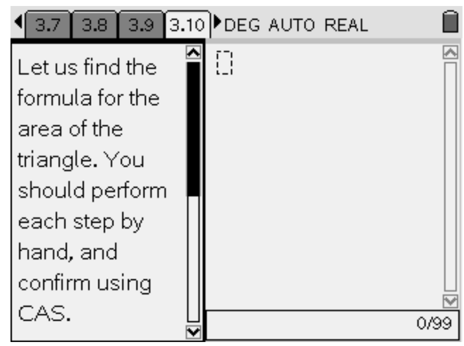
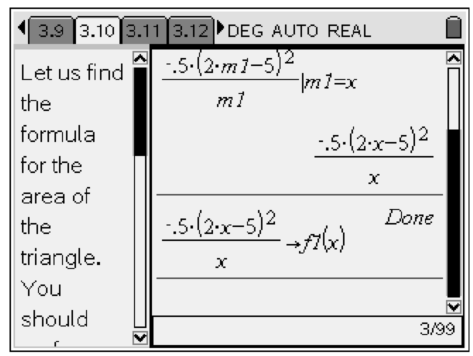
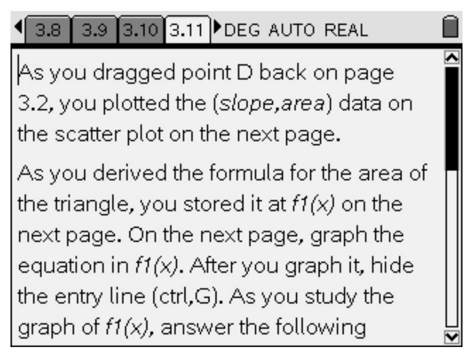
Page 3.9 at right shows an anticipate derivation of the y-intercept.	
Page 3.10 – The area formula. 1. Enter *.5(yint)(xint)* 2. Make the Substitution *m1=x*. 3. Store this expression at *f1(x).*	
Page 3.10 at right shows an anticipated derivation of the area formula, the substitution m1=x, and how it is stored at f1(x). *You should discuss with your students the need to substitute x for m1.*	
Pages 3.11 and 3.12 – Is our formula good? On these two pages, the formula derived and stored on page 3.10 is now graphed on top of the data. Students are asked to judge how well our formula fits the data through these guided questions: 1. How well does your area formula fit the data? How can you tell? 2. Pretend you are a bug crawling along this graph from left to right. What is happening to the values of the slope and the values triangle area as the bug crawls? 3. A part of this graph lies in the fourth quadrant. Does this part of the graph have any meaning in light of this triangle problem? Explain.	

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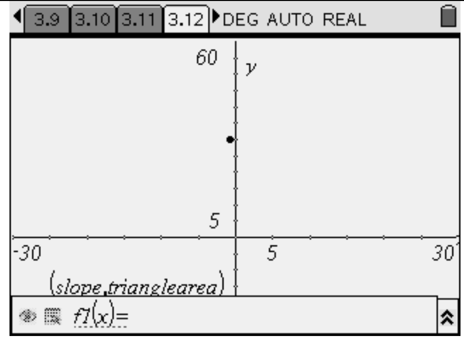
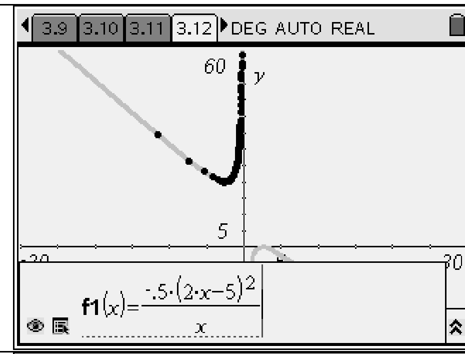
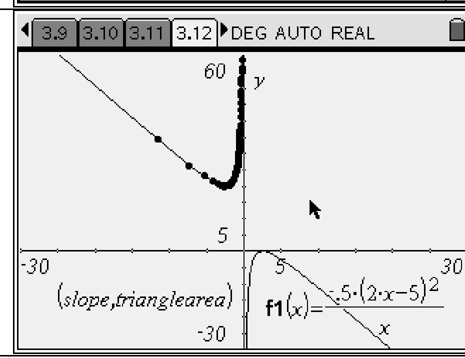


Page 3.12 at right shows the graph of the f1(x). The area formula will be stored at f1(x) from your work on page 3.10. You will need to press enter to graph the function.



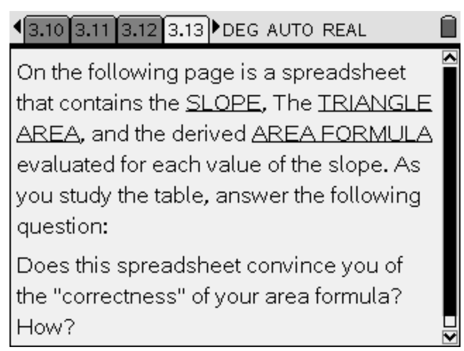
Page 3.12 at right shows the graph after hiding the entry line (ctrl,G).



Pages 3.13 and 3.14 – Is our formula good (part 2)?
On these two pages, students once again are asked to explore a spreadsheet that contains data on the slope, triangle area, and the derived area formula evaluated at each value of the slope.

There is only one guided inquiry question for this spreadsheet:

1. Does this spreadsheet convince you of the "correctness" of your area formula? How?



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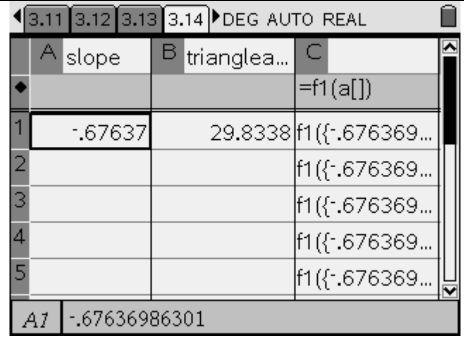
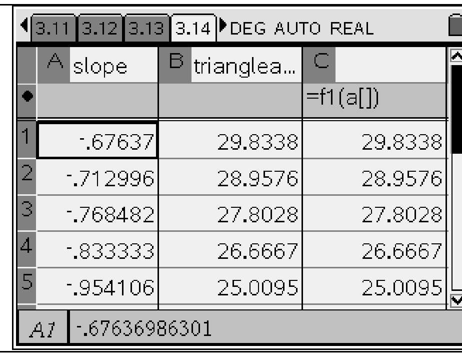


Page 3.14 at right shows the revised spreadsheet from page 3.4 with the function values of f1(x) displayed in column C.

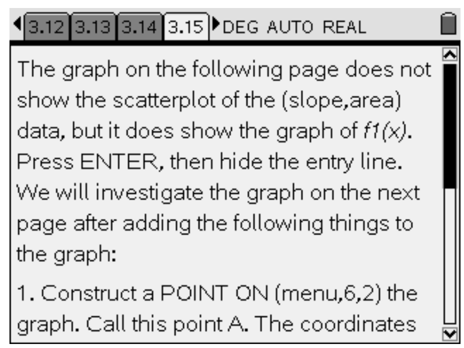


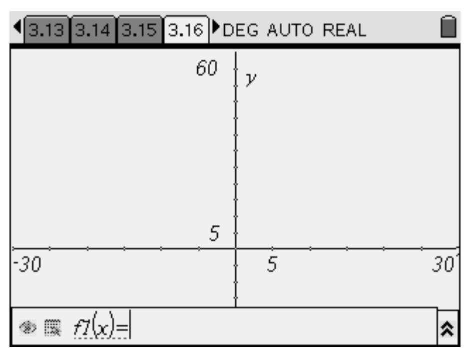
Pages 3.15, 3.16, and 3.17 – Use the formula to answer questions.
In these pages, students are guided to explore the graph of the area formula by tracing points along the graph and exploring changes in the values of the slope and area. Students first are asked on page 3.15 to construct some things on the graph to aide in the exploration of the graph:

2. Construct a POINT ON (menu,6,2) the graph. Call this point A. The coordinates of this point should be displayed.
3. Construct a line PARALLEL to the x-axis through the point on the graph (menu,9,2).
4. Construct the INTERSECTION POINT of this line and the graph (menu,6,3).
5. Find the COORDINATES of this intersection point (menu,1,6).

On page 3.17, students are asked the following guided inquiry questions:

1. Find the values of the slopes that produce an area of 38 square units. You can do this by editing the y-coordinate of point A.
2. Find the value(s) of the slope that produces the triangle with the minimum area by dragging point A to the apparent minimum on the graph. What are





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the values of the slope and area?

3. When point A is at the minimum point on the graph, is the parallel line you constructed TANGENT to the graph?
4. For all but one area value, there are two different slope values. Why?
5. When the triangle area increases by 1 square unit, by how much does the slope increase?
6. How do you explain these different increases?
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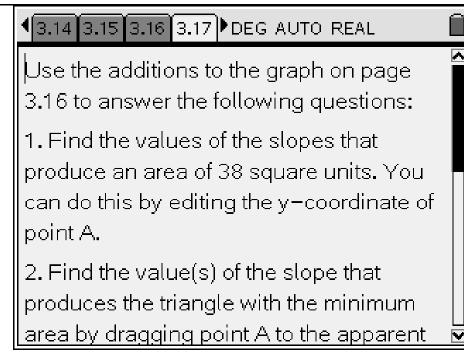

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<td>
Page 3.16 at right shows the graph of f1(x) with point A on the graph. When you first arrive at this page, you will need to press enter to graph f1(x), and then hide the entry line (ctrl,G). The coordinates of A have been moved to the upper right corner of the view screen.
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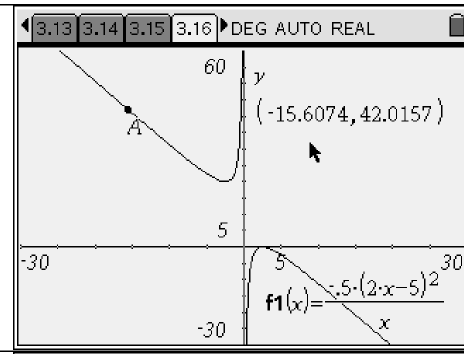

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<td>
Page 3.16 at right shows the parallel line constructed through A, and the other intersection of this line with the graph of f1(x).
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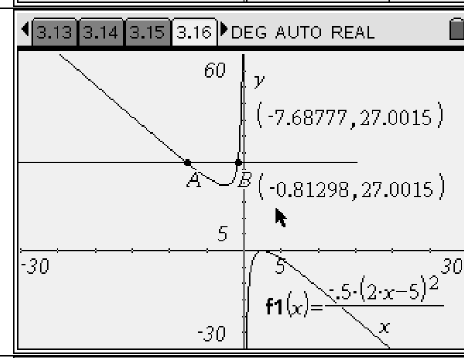

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Page 3.16 at right shows point A after dragging it to the apparent minimum on the graph. Notice the lower case m that appears, signifying the minimum on the graph. Notice that the coordinates for point B no longer appear, suggesting the parallel line is tangent to f1(x) at this point. The coordinates of the minimum point confirm what the students may already know; the minimum area is 20 square units, and the slope of the line is −2.5.
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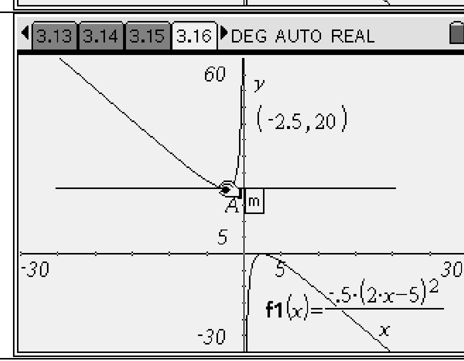

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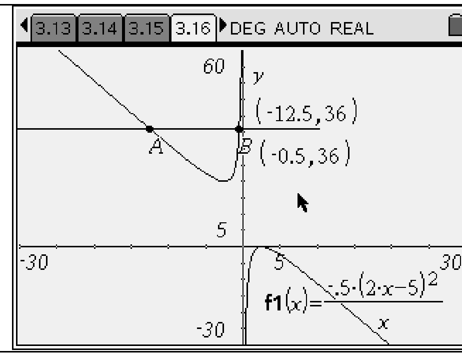
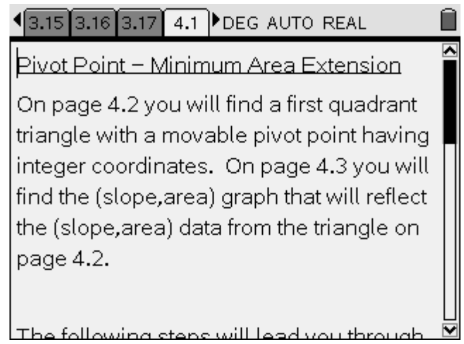
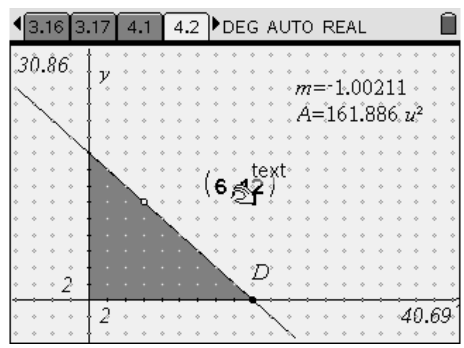
Page 3.16 at right shows the graph after editing the y-coordinate of point A to be 36. The coordinates of A and B may confirm what the students already know; slopes of −12.5 and −0.5 produce an area of 36 square units.



Problem 4 – Pivot Point and Minimum Area Extension.
On these optional pages, students are engaged in finding patterns among the coordinates of the pivot point and the area of the triangle, formulating conjectures about any relationships they discover, and ultimately proving these conjectures.

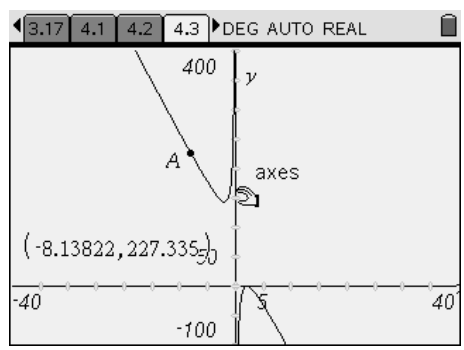


On page 4.2, students are asked to move the pivot point to another location and to drag point D to find the triangle with the minimum area. On page 4.3, the graph of the area function is automatically updated to reflect the coordinates of the pivot point. Students drag point A on the graph to the find the coordinates of the minimum (slope,area) value.



The steps for the investigation are as follows:

1. Move the pivot point on page 4.2, drag point D along the x-axis, and estimate the minimum area and the slope that produces this area.
2. On page 4.3, drag point A to the minimum point on the graph. What are the coordinates of the minimum point on the graph? You MAY need to adjust the window settings (menu,4,1) to see a good graph.
3. Repeat steps 1 and 2 until you can conjecture how the minimum area and corresponding slope is related to the coordinates of the pivot point.
4. Prove your conjecture.



The minimum area is twice the product of the coordinates of the pivot point. The slope that produces the minimum area is $\frac{y\text{-}coordinate}{x\text{-}coordinate}$.

Problem 5 – Pivot Point, Right Triangle, and Slope Investigation.
On these optional pages, students are engaged in finding patterns among the coordinates of the pivot point, the

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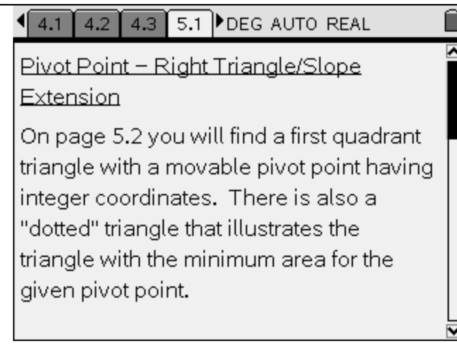
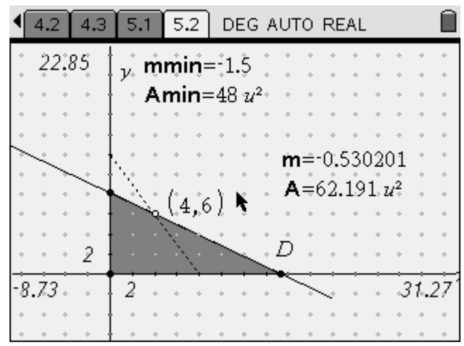
slope of the line and the area of the triangle. They are also looking at how the pivot point is related to the right triangle with minimum area. Students are formulating conjectures about any relationships they discover, and ultimately proving these conjectures.



On page 5.2 is the now familiar first-quadrant triangle. There is also a dashed segment showing the position of the first-quadrant triangle with minimum area. As students drag the pivot point to different locations, this segment will always show the triangle with minimum area.



The steps for the investigation are as follows:

1. Move the pivot point, and drag point D to the minimum triangle.
2. Observe the value of the slope, the minimum area, and the relationship of the pivot point to the minimum area triangle.
3. Repeat steps 1 and 2 until you can formulate a conjecture as to how the coordinates of the pivot point are related to the minimum area and to the corresponding slope, and how the pivot point is related in general to the minimum area triangle.
4. Prove your conjectures.

The pivot point is the Circumcenter of the right triangle with the minimum area. This means the pivot point is equidistant from the origin, the *x*-intercept, and the *y*-intercept.

