The Orthocenter of a Triangle

Definitions:

Orthocenter—the point of concurrence of the three altitudes of a triangle. Figure 1 shows othocenter H and the nine-point circle.

Nine-point circle, or **Feuerbach circle**, of a triangle—the circle centered at the midpoint between the orthocenter and the circumcenter of the triangle that passes through the midpoints of the sides (three points), the feet of the altitudes (three points), and the midpoints of the segments connecting the vertices to the orthocenter (three points).

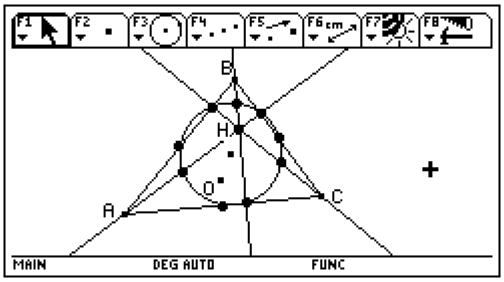

Figure 1

Construct and Investigate:

1. Draw and label $\triangle ABC$ on the Voyage™ 200. Construct the altitudes, and locate the orthocenter of the triangle. Would you need to construct all three altitudes? Explain why or why not.
2. Hide the altitudes, and drag the triangle around the Voyage 200 screen. Explain where the orthocenter is located for different types of triangles.
3. Hide the original triangle so that only the three points A, B, and C (representing the vertices), and the orthocenter H remain on the screen. Drag the vertices around the screen, and investigate the relationship of these four points to one another. Test your conjectures.
4. How many ways can three items be chosen from a collection of four items if order is not important and there is no replacement? Choose all the subsets of three of the four points on the screen, and construct all possible triangles defined by these four points. Draw the nine-point circle for each of the triangles you constructed. (**Hint:** Make a macro after the first one is constructed.) Drag the original triangle around the screen. Explain what happens. (**Hint:** Point at the nine-point circle with the **Pointer** arrow. What happens?)
5. Using a macro, construct the circumcircles of the triangles in your figure. What is the relationship of these circles to each other? Explain. What is the relationship between these circumcircles and the nine-point circle? Explain.
6. Hide the circumcircles and the nine-point circles, and connect the circumcenters of the four triangles in this figure. Investigate the relationships between the original figure and the one formed by connecting the circumcenters. Explain the relationships you find.
7. Construct the nine-point circle for the new figure formed in part 6 above. How does this compare to the nine-point circle for the original figure? Explain.

Explore:

1. Draw and label a new acute triangle, and construct its orthocenter with three altitudes. Draw the **orthic triangle**, the triangle with vertices located at the feet of the altitudes of the original triangle. Explore the relationships between the original triangle and the orthic triangle by investigating the relationships between their incenters, incircles, circumcenters, circumcircles, orthocenters, and nine-point circles. Explain what you find and how all the different centers and circles are related.
2. The lines representing the sides of a bisected angle are called the **isogonal lines**, and each one is called the **isogonal conjugate** of the other. By this definition, adjacent sides of a triangle are isogonal conjugates of each other. Investigate the angular relationship between the orthocenter, the incenter, the circumcenter, and a vertex of a triangle. Explain what you find.

Teacher's Guide: The Orthocenter of a Triangle

Construct and Investigate:

1. The altitudes of a triangle intersect at the orthocenter. The intersection of any two of the three altitudes locates the orthocenter because two nonparallel lines in a plane determine a unique point of intersection. It is interesting that all three altitudes are coincident at the same point, but once this is established, only two altitudes are needed to locate the orthocenter.
2. When the triangle is acute, the orthocenter is located in the interior of the triangle because the feet of the altitudes lie on the sides of the triangle. When the triangle is obtuse, the orthocenter is exterior to the triangle because two of the altitudes fall outside the triangle. When the triangle is a right triangle, the orthocenter is located at the vertex of the right angle because two of the altitudes of a right triangle are the legs of the right angle.
3. One of the most beautiful symmetries of a triangle is represented by the relationship of the **orthic set of points** made up of the vertices of a triangle and its orthocenter. The triangle formed by any combination of three of these points has the fourth point as its orthocenter (Figure 2).

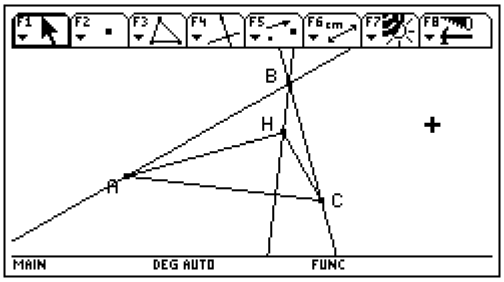

Figure 2

4. The **combination** of four items taken three at a time is given by the equation:

   $${}_4C_3 = \frac{4*3*2}{3*2*1} = 4.$$

   Four triangles are possible using this set of four points. The nine-point circles for all four triangles are the same (Figure 3). By definition, the nine-point circle of a triangle passes through the feet of the altitudes, the midpoints of the sides, and the midpoints of the segments joining the vertices to the orthocenter of the triangle. Because each point in the orthic set is the orthocenter of the triangle formed by the other three, the sides of each triangle lie on the altitude of another triangle. The midpoint of each side of a triangle is also the midpoint between a vertex and an orthocenter. Because the same nine points are interchangeable, all four triangles have the same nine-point circle.

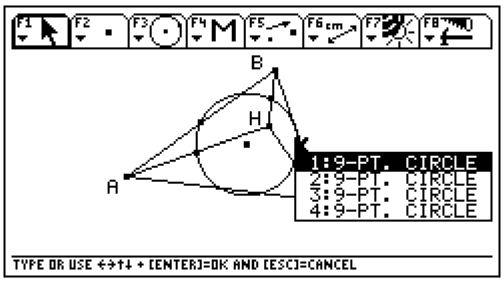

Figure 3

   An interesting extension of this idea is to find the **centroids** of the four triangles formed by the orthic set of points. These centroids also form an orthic set with all the same properties of the orthocenters (Figure 4).

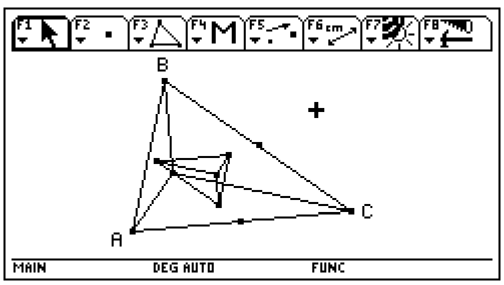

Figure 4

The figure formed by the orthic set of centroids is similar to the original figure, but the segments are only one-third the length of the corresponding segments in the original figure; the corresponding areas are only one-ninth as large as the original (Figure 5).

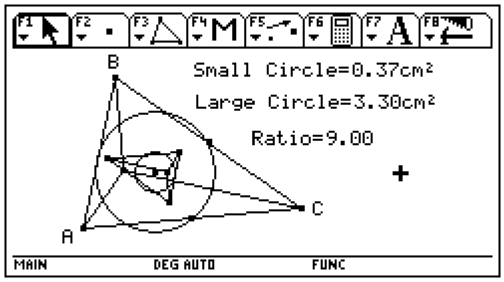

Figure 5

This area comparison includes the corresponding circumcircles and nine-point circles. The nine-point circles of two figures are concentric. This means that the original figure can be transformed onto the smaller figure by a rotation about the center of the nine-point circle, followed by a one-third dilation.

5. The areas of the four circumcircles are equal; hence, the circumcircles are congruent (Figure 6).

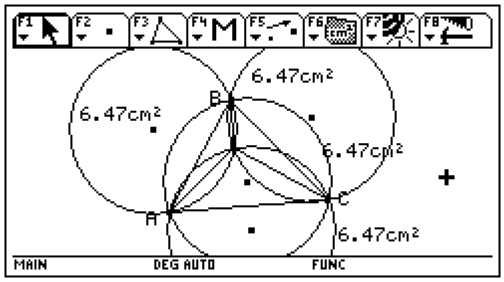

Figure 6

The area of the nine-point circle is one-fourth the area of any one of the circumcircles. This is a property of any nine-point circle when compared with its corresponding circumcircle (Figure 7).

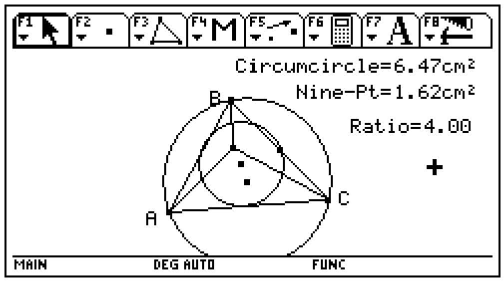

Figure 7

6. When all of the circumcenters are connected to all the other circumcenters, a figure congruent to the original figure is formed (Figure 8).

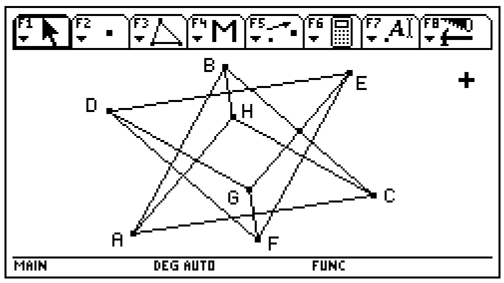

Figure 8

The circumcenters of the original figure, ***D***, ***E***, ***F***, and ***G***, have become the orthocenters of the new figure. This means that the orthocenters of the original figure, ***A***, ***B***, ***C***, and ***H***, are the circumcenters of the new figure.

7. Any triangle in the new figure has the same nine-point circle as any triangle in the original figure (Figure 9).

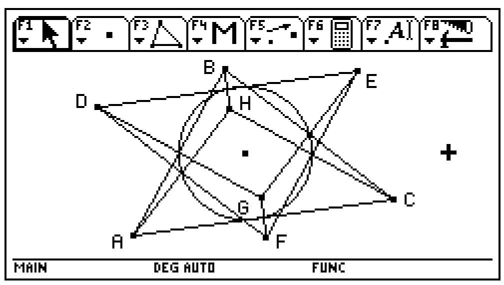

Figure 9

The center of the nine-point circle is the midpoint of the segment connecting the orthocenter to the circumcenter. Because the orthocenters and circumcenters of these two figures are interchangeable, the center of the nine-point circle represents the center of rotation that transforms the original figure into the new figure.

Explore:

1. Figure 10 shows △***ABC*** with orthocenter ***H*** and **orthic triangle** △***DEF***. The orthic triangle represents the inscribed triangle with the smallest perimeter.

   For more information on this property, see "Fagnano's Problem" in *Explorations for the Mathematics Classroom* (Vonder Embse and Engebretsen, 1994, p. 36).

   The angle bisectors of the orthic triangle are the altitudes of △***ABC***; therefore, the orthocenter ***H*** is also the incenter of △***DEF*** (Figure 10).

   The nine-point circle of △***ABC*** is the circumcircle of the orthic triangle △***DEF***. This nine-point circle is four times the area of the nine-point circle of △***DEF*** and one-fourth the area of the circumcircle of △***ABC*** (Figure 11).

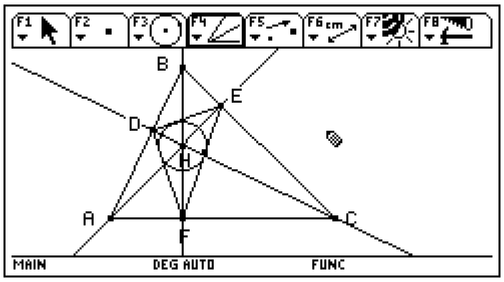

Figure 10

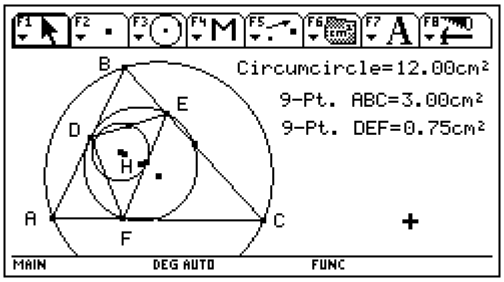

Figure 11

2. For △***ABC*** shown in Figure 12, △***HBO*** is formed by the circumcenter ***O***, a vertex ***B***, and the orthocenter ***H***. It is bisected by the line through the vertex ***B*** and the incenter ***I***.

   In Figure 12, segments $\overline{HB}$ and $\overline{OB}$ are isogonal conjugates of each other because line ***BI*** bisects △***HBO***. This relationship is true for each vertex of the triangle.

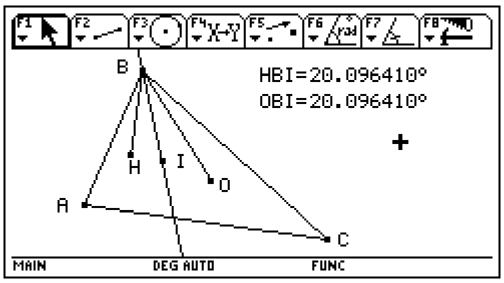

Figure 12

