NUMB3RS Activity: Thinking Backward
Episode: "Pandora's Box"

Topic: Inverse problems **Grade Level:** 9 - 12
Objective: Solve two classical inverse problems.
Time: 30 minutes
Materials: TI-83 Plus/TI-84 Plus graphing calculator (optional)

Introduction

In "Pandora's Box," Charlie uses a wavelet-based *deconvolution algorithm* to "unsmear" a smudged fingerprint. A deconvolution algorithm analyzes the picture, sharpens the edges, and restores the details—thus finding the picture beneath the smear. Finding such an algorithm to determine a clear image of a fingerprint is an example of solving an *inverse problem*. In this activity, students consider two classical inverse problems after considering a simulation of a deconvolution algorithm on the graph of a function.

Discuss with Students

The basic idea of deconvolution is this: given a blurred image (e.g., a photograph or fingerprint), how can you construct a clear (unblurred) version of the image? One method is to determine a deblurring function from an equation of the form $g = H(f) + n$, where g is the blurred image or blurring function, H is the distortion operator (also called the point-spread function [PSF]), f is the original clear image or deblurring function, and n is "noise" that further corrupts the image. The particular combination of H and f creates the distortion. Given g and some information about H and n, the goal is to find f by "running this formula backwards." This process is very complicated, but it is one example of solving an *inverse problem*.

Roughly speaking, inverse problems can be described as problems where the answer is known, but the question is not. In an inverse problem, we want to determine the internal structure of a system from external measurements or observations. In direct problems, we find *effects for causes*. In inverse problems, we find *causes for effects*.

One example occurs in medical imaging. The direct problem is: if the exact properties of some internal organ were known, then you would know the images of the organ determined by the scan. However, it is almost always true that you are trying to find the properties of the internal organ. Thus, there is the inverse problem: Given the images determined by the scan, infer the properties of the organ.

Some of the ideas for this activity were adapted from <u>Inverse Problems</u> by Charles Groetsch (See Extensions for more details).

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Student Page Answers:

1a. *Answers vary: parabola is common response.* **1b.** *Answers vary: parabola of the form* $y = ax^2 + 1$ *is a common response* **1c.** *Answers vary:* $y = x^2 + 1$ *is common response.* **1d.** *No, the graphs of* $y = \frac{|x^3|}{2} + 1$ *and* $y = \frac{(e^{1.15x} + e^{-1.15x})}{2}$ *are very similar to the one shown.* **2a.** *There are 7 solutions: as a sum of 3, 4, 5, 7, 12, 15, and 20 positive integers.* **2b.** *Because* $1 + 2 + \cdots + 20 = 210$, *the number of solutions n satisfies* $n \leq 20$*; check cases; OR* $(k + 1) + (k + 2) + \cdots + (k + n) = 210 \rightarrow kn + n(n + 1) \div 2 = 210 \rightarrow n(2k + n + 1) = 420$ *so n is a factor of 420 for which n and* $420 \div n$ *have opposite parity.* **3.** four; one on each side of the table **4.** $P = (13/3, 0)$; $\alpha = \tan^{-1}(3/2) \approx 56.3°$
5. $P = (10, 3.4)$; $\alpha = \tan^{-1}(5) \approx 78.7°$

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Name: ________________________________ Date: _______________

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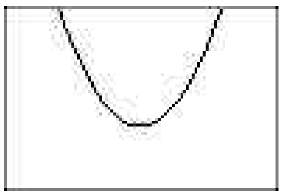
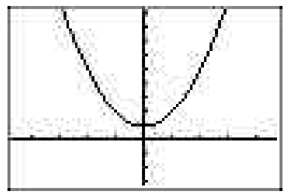
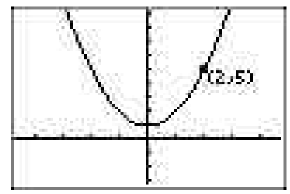
In "Pandora's Box," Charlie uses a wavelet-based *deconvolution algorithm* to "unsmear" a smudged fingerprint. A deconvolution or "deblurring" algorithm analyzes the picture, sharpens the edges, and restores the details—thus finding the picture beneath the smear.

These three figures simulate the image of the graph of a function becoming clearer as a "deblurring" algorithm is applied.







1. a. Looking only at the first image (on the left), what can you say about the function?

b. Looking only at the first two images (left and center), what can you say about the function?

c. Looking at all three images, what can you say about the function?

d. Can you positively identify the function after looking at all three images? Why or why not?

Finding a "deblurring" function is an example of solving an *inverse problem*. Roughly speaking, inverse problems could be described as problems where the answer is known, but the question is not. In direct problems, we find *effects for causes*. In inverse problems, we find *causes for effects*. Typically an inverse problem has more than one solution.

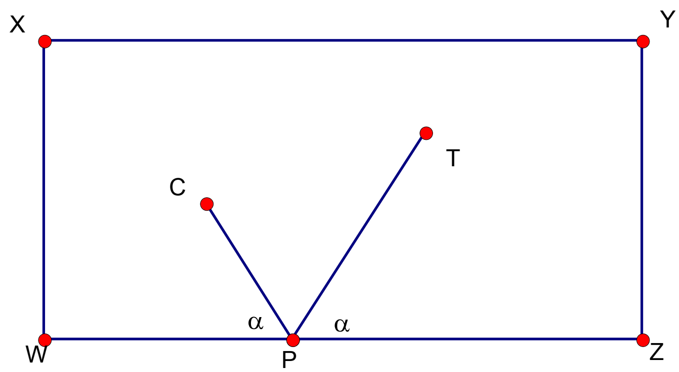
2. Consider the direct problem, "Find the sum 40 + 41 + 42 + 43 + 44."

a. One inverse problem is: "Find all ways to write 210 as the sum of consecutive positive integers." How many solutions does this problem have?

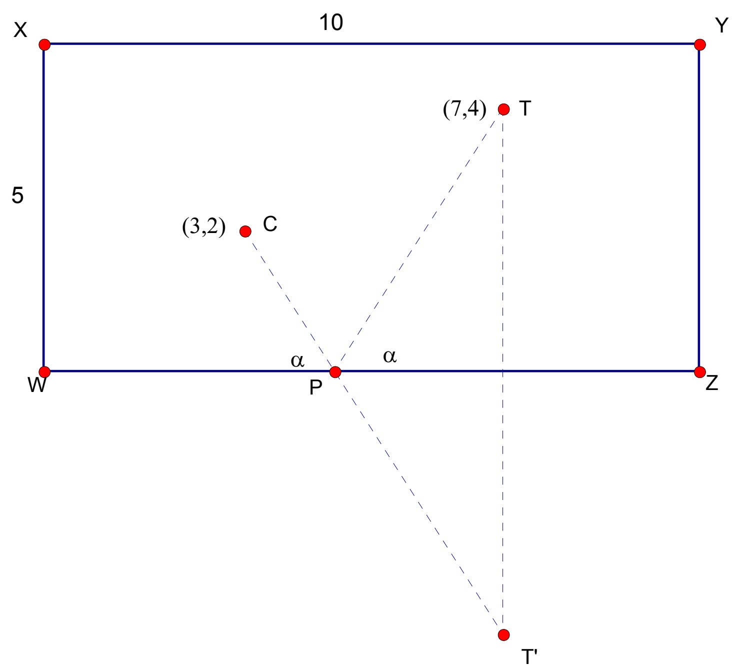
b. How do you know that you have found all of the solutions?

The *scattering problem* is one of the most important inverse problems in science.
A signal of some type is transmitted, strikes an object (the scatterer), and is bounced off the object (scattered). The scattered signal is then collected, and the characteristics of the scatterer are inferred from the information contained in the scattered signal. Familiar applications include radar, sonar, and medical imaging.

A simplified version of an inverse scattering problem is the billiard ball problem. When you hit the ball located at C while aiming at P, you expect the ball to go to T. Thus, the direct problem is to determine the path of the ball from C to T, knowing the impact point P, which also determines the angle of incidence α. An inverse problem is: Given the location of the ball before (C) and after (T) you hit it, determine the impact point and the angle of incidence using the Reflection Principle (sometimes attributed to Heron). The points C and T make up the "scattered signal," while the point P (and angle α) is the "scatterer."



Consider the billiard ball situation given in the figure. The table is 5 feet wide and 10 feet long. Assume there is a coordinate system with origin W so that Z = (10, 0) and X = (0, 5).
One inverse problem is: Given the beginning and ending positions of the ball as C = (3, 2) and T = (7, 4), find the coordinates of an impact point on one side P and the angle of incidence α.



To determine a possible location of P using the Reflection Principle, reflect T in one of the sides and note its location T′. Draw the segment CT′. The intersection of CT′ and side WZ is an impact point P. Then $m\angle CPW = m\angle TPZ = \alpha$.

3. How many locations for P are possible?

4. Find the coordinates of point P and the measure of angle α if the ball bounces off side WZ as shown in the figure.

5. Find the coordinates of the impact point and its associated angle of incidence if the ball only bounces off side YZ to get to T.

The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

- In the situation with C = (3, 2) and T = (7, 4), find the coordinates of the remaining impact point P and the associated angles of incidence.
- In the situation with C = (3, 2) and T = (3, 4), find the coordinates of all possible impact points and the associated angles of incidence.
- It is possible for a ball to hit more than one side as it goes from one point to another.

  a. Suppose C = (3, 2) and the ball hits all four sides in the order WZ, YZ, XY, XW as it goes from C back to C. What is the total distance traveled by the ball?

  b. In general, suppose C = (x, y) and the ball hits all four sides in the order WZ, YZ, XY, XW as it goes from C back to C. What values are possible for the total distance traveled by the ball?
- The method Eratosthenes (276BC – 194BC) used to estimate the circumference of the Earth is a classic inverse problem. Find out about his method, and discuss why it is an example of an inverse problem.
- Finding the roots of a quadratic equation is a direct problem. An associated inverse problem would be finding a (monic) quadratic equation given its roots. Find some other examples of such pairs of direct-inverse problems in your mathematics course.

Additional Resources

- For more information on how the inverse problem is used in solving crimes, read the article “Crime Fighting Maths” by Chris Budd at **http://plus.maths.org/issue37/features/budd/index-gifd.html**
- The deconvolution algorithm and inverse problem are also being used in modern technology. To find out more, read the article “Hiding Messages in Plain Sight” at **http://news.bbc.co.uk/2/hi/technology/6361891.stm**.
- For more information on inverse problems, read the following book: Groetsch, Charles. *Inverse Problems*. Washington DC: The Mathematical Association of America,1999.

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