NUMB3RS Activity: Perfect Out-Shuffles
Episode: "Double Down"

Topic: Card shuffling **Grade Level:** 7 - 12

Objective: Review and apply knowledge of functions to a non-routine problem

Time: 30 - 75 minutes, depending on how much material is used.

Materials: Index cards, graphing calculator

Introduction

In "Double Down," card auto-shufflers in casinos are discussed. These devices can be used with more than one deck of cards, and offer casinos a method for randomizing the cards that are dealt, while minimizing any opportunity for the person dealing the cards to cheat. However, there are ways to manually deal cards in a predetermined manner.

Discuss with Students

The focus of the activity is on a manual, hands-on method of shuffling a pile of cards, known as the *perfect out-shuffle* or the *Faro shuffle*. The arrangement of the cards after one perfect shuffle is not random, because the position of any card can be determined mathematically without actually doing the shuffle. This shuffle is intriguing because after repeated shuffling, the order of the cards will always return at some point to the original order. The phrase "at some point" leads to the challenge of determining the relationship between the number of shuffles needed to return a deck of cards to its original order and the number of cards in the deck. Magicians use perfect shuffles to perform a large number of stunning magic tricks. Ask your students why they think magicians would be interested in this type of shuffle.

Instructions are given in the activity for performing a perfect out-shuffle. Students must follow the instructions exactly as they are written. Note that the very first card that is set on the table must come from the *bottom* of the pile of cards in the left hand.

Student page answers: **1.** *$x = 1$, $y = 1$; $x = 2$, $y = 3$; $x = 3$, $y = 5$; $x = 4$, $y = 7$; $x = 5$, $y = 2$; $x = 6$, $y = 4$; $x = 7$, $y = 6$; $x = 8$, $y = 8$*

2. *The equation for y in terms of x is a piecewise function defined as follows: $y = 2x - 1$ for $1 \le x \le 4$ and $y = 2x - 8$ for $5 \le x \le 8$ (Note: the domain of the function is $x = \{1, 2, 3, 4, 5, 6, 7, 8\}$*

3.

x	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
f(x)	*1*	*3*	*5*	*7*	*2*	*4*	*6*	*8*
f(f(x))	*1*	*5*	*2*	*6*	*3*	*7*	*4*	*8*
f(f(f(x)))	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*

4a. *1* **4b.** *2* **4c.** *4* **4d.** *5*

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Name: ________________________________ Date: _______________

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In "Double Down," card auto-shufflers in casinos are discussed. In this activity, you will analyze a different type of method for shuffling cards that is not random in nature. This shuffle is used by magicians to mix a deck of cards in a predetermined way, and is commonly called a *perfect out-shuffle* or a *Faro shuffle*. To begin, try doing this shuffle with 8 cards.

Step 1: Obtain 8 index cards and write the letters A, B, C, D, E, F, G, and H on the cards as shown below (one letter per card).

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**

Step 2: Arrange the 8 cards in alphabetical order in a pile, with card A at the top and card H at the bottom. Pick up the 8 cards and hold them in your left hand.

Step 3: Take the top four cards in your right hand and leave the bottom four cards in your left hand.

Step 4: Place the bottom card of the pile in your left hand on the table. Then place the bottom card of the pile in your right hand on top of the card on the table.

Step 5: Repeat step 4 until all eight of the cards are in a pile on the table.

If you have done the shuffle correctly, the order of the 8 cards from top to bottom is **A, E, B, F, C, G, D, and H**. Notice that the cards that were in the top and bottom positions (the outside cards) of the original stack of cards are still in the top and bottom positions of the shuffled stack of cards. This is why this shuffle is called a perfect out-shuffle. (The word perfect is used because the cards are being moved in a predetermined manner instead of randomly.)

If you do this shuffle two more times, the cards will be in the same order as they were at the start! That is, after only three perfect out-shuffles, a deck of 8 cards will return to its original order.

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1. Look at Table 1 below. The before column displays the order of the 8 cards before being shuffled, and the after column shows the positions of the cards after one shuffle.

Table 1	
Before	**After**
A	A
B	E
C	B
D	F
E	C
F	G
G	D
H	H

Table 2	
x	***y***
1	
2	
3	5
4	
5	
6	
7	
8	

Let x represent the position of a card before the shuffle and the y represent the position afterwards. For example, note that the card in the third position (the C) ended up in the fifth position. This can be described by saying that if $x = 3$, then $y = 5$. Use the information in Table 1 to complete the missing entries in Table 2.

2. Using Table 2, develop a formula that can be used to produce the value of y for all values of x between 1 and 8. (Hint: You will need to use a piecewise function. Look at the y-values for $1 \le x \le 4$ and $5 \le x \le 8$.) How would your formula change if a pile of 52 cards were used instead of 8?

3. Let $y = f(x)$, where f is the function that describes the relationship between y and x you found in Question #1. Find each of the following.

x	1	2	3	4	5	6	7	8
$f(x)$								
$f(f(x))$								
$f(f(f(x)))$								

4. At the start of this activity we saw that the order of a deck of 8 cards will be restored after 3 perfect out-shuffles. Using some index cards, make the following decks. For each deck, manually perform as many perfect out-shuffles as needed to restore the cards to their original order. Search for a pattern and then predict how many perfect shuffles would be needed for a deck of 64 cards to its original order.

(a) 2 cards (b) 4 cards (c) 16 cards (d) 32 cards

The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

In this extension, you will have the opportunity to study the relationship between the number of cards in a deck n (where n is not a power of 2, as was the case in Question #4) and the number of shuffles S required to restore the deck to its original order.

You will need to install the program SHUFFLE onto your TI-83/84 Plus calculator. This program can be downloaded for free at **http://education.ti.com/exchange**. Once the program has been loaded, run the program SHUFFLE. This program performs a perfect out-shuffle of n cards, and continues to shuffle the cards until the original order has been restored. The number of shuffles is displayed on the home screen at the end.

Use the SHUFFLE program to complete the tables below.

Number of Cards	Number of Perfect Shuffles
2	
4	
6	
8	
10	
12	
14	
16	
18	
20	
22	
24	
26	
28	
30	

Number of Cards	Number of Perfect Shuffles
32	
34	
36	
38	
40	
42	
44	
46	
48	
50	
52	
54	
56	
58	
60	

Using your completed tables, look for a relationship between the number of cards in the deck and the number of perfect shuffles required to restore the deck to its original order. How many perfect out-shuffles would be needed to restore decks of 62 or 66 cards to their original orders? Use the SHUFFLE program to check your predictions.

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Shuffling and Magicians

- When magicians perform a perfect out-shuffle, they do not perform it using the method described in the activity. Skilled magicians are able to do the entire shuffle while holding the cards in their hands, without using a table. Some magicians are able to do this shuffle with one hand, a feat that leaves people stunned and in awe of their skill. Details about how the deck is held and what the magician actually does can be found at **http://web.superb.net/cardtric/sleights/outfaro.htm**.

- Another type of shuffle is called a *perfect in-shuffle*. To perform this shuffle, the first card you put on the table should be the bottom card from the pile of cards in your *right* hand. Using what you know about perfect out-shuffles, make a prediction about what the result of a perfect in-shuffle would be (think about the positions of the top and bottom cards of the original deck as well as the positions of the top card of each half of the deck). Then check your prediction by using the 8 cards you created for the activity to perform a perfect in-shuffle. What do you notice about the order of the cards? How might a magician use a perfect in-shuffle?

- Magicians most typically use a standard deck of 52 cards. Why might a magician want to perform a perfect out-shuffle? Think about how the magician could take advantage of the fact that the position of the top and bottom cards never change. Also, how might a magician take advantage of the fact that the cards can be secretly restored to their original positions?

Additional Resources

Magic Tricks, Card Shuffling and Dynamic Computer Memories, S. Brent Morris
ISBN 0-88385-527-5, Mathematical Association of America, **www.maa.org**

The Mathematics of Games, John D. Beasley, Oxford University Press,
ISBN 0-19-286107-7

Ivars Peterson's Math Trek - Magic of Perfect Shuffles - August 1, 1998
http://www.sciencenews.org/sn_arc98/8_1_98/mathland.htm

Shuffle: **http://mathworld.wolfram.com/Shuffle.html -** includes links to faro shuffle, perfect shuffle, and others.

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