NUMB3RS Activity: Have Your Cake and Eat It Too
Episode: "One Hour"

Topic: Discrete Math/Fair Division **Grade Level:** 9 - 12
Objective: Students will explore an algorithm that evenly divides resources
Time: 10 - 20 minutes

Introduction

In "One Hour," the FBI is chasing kidnappers who demand a ransom of $3.2 million. The FBI thinks that this number has a personal significance to the kidnapper, but Charlie disagrees. He compares the division of the money among the crew to the distribution of a cake among a group. Charlie explains that if two people agree to divide the cake, "the obvious strategy is to cut it down the middle. But if one person likes frosting more than the other, one person prefers vanilla cake or one person prefers chocolate cake with vanilla frosting, it may be that slices unequal in size actually create a fairer distribution." Charlie uses cake-cutting algorithms to conclude that the head kidnapper's goal is to get $1.65 million, an amount with a significant meaning to one person involved. In this activity, students explore fair ways for two people to share a cake and then extend the exploration for three people.

Discuss with Students

The activity begins with the "I cut, you choose" method to divide a cake between two people. One person makes a cut to divide the cake into what he or she considers to be two equal parts; the other person chooses the first piece. To get your students to think about this, ask them how they think two people could agree to fairly divide a cake.

Student Page Answers:

1a. *Cut on the far left side; the right side is the entire cake.* **1b.** *Cut on the far right side; the left side is the entire cake.* **1c.** *As Charlie slides the knife over the cake from left to right, his value of the left side increases from 0% to 100%. Because the cake is one continuous piece with an infinite number of possible cuts, there must be some cut that values the left side at 50%. The right side will also be 50% for this cut.* **2.** *Charlie has already stated that the two pieces are of equal value, so he will be happy with either piece. If Don considers the pieces of equal value, he can take either and will be happy. If he feels one is worth more than the other, he can take that one and Charlie still gets a piece he considers equal in value to Don's. Notice that Don has actually received what he considers to be more than a fair share*.
3a. *This method is not fair. Amita chooses last, so she has incentive to cut equal pieces. Charlie will pick the piece that he favors the most, so Don may be forced to choose between two pieces that he does not consider to be 1/3 of the original.* **3b.** *This method is fair, though cumbersome and potentially messy and less satisfying than an intact piece of cake.* **3c.** *This method is fair and can be extended to any number of people.* **4.** *Answers will vary.*

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Name ____ Date _______________

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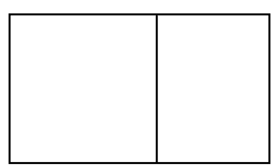
Consider the case of two people who agree to fairly divide a cake. A fair share means that both people consider the piece they receive to be at least half the value of the cake. As Charlie mentions in the episode, people can have different opinions about the value of a piece. For example, someone who likes frosting may put a high value on a piece of cake with an abundance of frosting, even if it has less volume.

One person makes a cut to divide the cake into what he or she considers to be two equal parts; the other person chooses the first piece. This is called the "I cut you choose" method. The first person is likely to cut the pieces as equally as possible, because he or she could otherwise be stuck with a less desirable piece.

Charlie and Don are using the "I cut, you choose" method to divide a cake. Charlie will make one vertical cut. Each cut will create two pieces, one on the left and one on the right. Charlie will consider each piece to be worth a percentage of the whole. For example, for the cut shown below, he may feel that the left side is worth 40% of the cake and the right side is worth 60% of the cake. (Remember that Charlie is making a judgment based on his personal preference, not necessarily on dimensions.)



1. **a.** Where has Charlie placed the cut if he considers the left piece to be 0%?

b. Where has Charlie placed the cut if he considers the left piece to be 100%?

c. Explain why there must be one cut that Charlie can make so that he values the two pieces equally.

The answer to Question 1c involves the Intermediate Value Theorem, a concept explored in calculus. This theorem states that for a function f that is continuous on the interval $[a, b]$, if there exists a value d between $f(a)$ and $f(b)$, then there is a value of c in (a, b) such that $f(c) = d$. For example, if someone was 5 feet tall last year and is now 5 feet 2 inches tall, at some point that person was 5 feet 1 inch tall. With the same reasoning, there is at least one place a person can cut a cake to create two pieces of equal value. The trick, of course, is to find that place. The

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Intermediate Value Theorem is called an existence theorem, because it guarantees that a value exists, but does not give a method to find that value.

In the two-person scenario, the first person determines where to cut the cake. Charlie cuts the cake into what he considers two pieces of equal value. Don will choose either piece, and Charlie will receive the other.

2. Explain why the brothers both receive a piece they believe to be at least 50% of the cake.

How does the scenario change if three people are trying to fairly divide a cake? A fair division in this case means each person believes that he or she receives at least a third of the value of the cake.

3. Decide if each method below is fair.

a. Amita cuts the cake into what she considers to be equal thirds. Charlie chooses the first piece, Don chooses second piece, and Amita receives the remaining piece.

b. Charlie cuts the cake into what he considers equal halves. Amita chooses one piece and Charlie keeps the other. Amita and Charlie each cut their piece into what they consider to be equal thirds. Don chooses one piece from Charlie and one piece from Amita. Charlie and Amita keep what is left.

c. Amita indicates a piece she considers to be worth a third of the value of the cake. Charlie and Don have the opportunity to inspect the piece. If either one thinks the piece represents more than a third, he indicates how to trim the piece so the value, in his opinion, is only one-third of the whole. The portion is given to the person whose cut is accepted by the other two people. After the first piece is taken, the remaining two people use a similar process on what remains of the cake.

4. Develop your own method for fairly dividing a cake between three people.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

For the Student

- Consider again the three-person scenario and the following fair division question: can the cake be cut into three pieces and allocated so that every person believes he or she receives the most desirable piece? The surprising answer is yes. The proof gives the following two possibilities. For each case, all three people have independently submitted a description of preferences. Using these preference diagrams, the cake has been cut strategically into pieces 1, 2, and 3. You can read more detail on the preference diagram solution and its application to other areas such as division of land and money in "The Heart of Mathematics," listed below.

  Case 1: Person A values all three pieces equally. Person B values pieces 1 and 2 equally and values piece 3 least of all. Recalling that person C's preferences are also known, explain how everyone can receive what they consider the best of the three pieces.

  Case 2: Person A values pieces 1 and 2 equally and values piece 3 least of all. Person B values pieces 2 and 3 equally and values piece 1 least of all. Recalling that person C's preferences are also known, explain how everyone can receive what they consider the best of the three pieces.

- One potential problem with "I cut, you choose" is the person who cuts may think that the other person has received more than what he or she considered 50%. A way to compensate for that potential involves a concept called the Pareto Optimality. Investigate this idea by going to the following Web sites:
  http://www.econlib.org/library/Enc/bios/Pareto.html
  http://www-cdr.stanford.edu/NextLink/papers/pareto/pareto.html

- Research the allocation of land in Germany after World War II. Some details can be found in "For All Practical Purposes," listed below.

Additional Resources

- Burger, Edward B, and Michael Starbird. *The Heart of Mathematics: An Invitation to Effective Thinking*. 2nd ed. Emeryville, CA: Key Curriculum Press, 2004.

- Consortium for Mathematics and Its Applications (COMAP). *For All Practical Purposes*. 6th ed. New York: W. H. Freeman & Company, 2003.

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