EXPLORATIONS™

Computing Indefinite Integrals

by

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Objectives

- To use transformations to remove integral symbols
- To simplify expressions

Materials

- TI-89 or Voyage™ 200
- Symbolic Math Guide (SMG) handheld application

Computing Indefinite Integrals

Activity Overview

The example problems in this activity provide ideas for using Symbolic Math Guide (SMG) to help you learn to solve indefinite integration problems. The key transformations in SMG are substitution and integration by parts. SMG also includes partial fractions, sum/difference and scalar product transformations as well as transformations for indefinite integrals of basic functions.

Setting Up the TI-89 or TI Voyage™ 200

Before starting this problem, make sure that the TI-89 or TI Voyage™ 200 has SMG installed. If the TI-89 or TI Voyage™ 200 does not have SMG version 2.00 or higher installed, go to education.ti.com to download the free app. If you are not familiar with SMG, the SMG User Guide and the SMG Guided Tour are available at the same link.

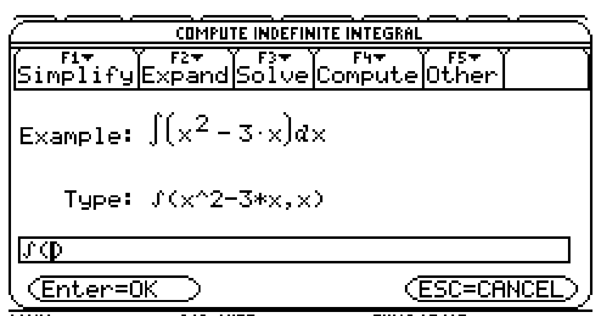
Start SMG by pressing [APPS], choosing SMG, and then pressing [ENTER].

Entering Indefinite Integration Problems

Example

Enter $(2x+1)^{25}\,dx$.

1. Press [F2] **1** and then press [F4] **2.** The display should be similar to the one shown at the right.



2. Type **(2x + 1)^25,x.** Press [ENTER].

The Indefinite Integrals of Basic Functions

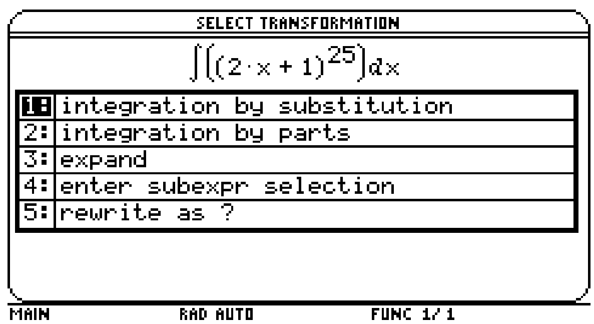
You learn to compute indefinite integrals by learning to recognize the antiderivatives of functions such as polynomials, sine, cosine, tangent, ln, log, and so forth. Your teacher may construct a problem set of basic indefinite integral problems and provide them to you to solve using SMG. See Student Reference Sheet 1, **Basic Indefinite Integrals**, for a table of some of the basic indefinite integrals that SMG knows. In addition, your teacher may choose examples from textbooks or other references for you to solve. An example set of basic indefinite integral problems is included in Student Work Sheet 1, **Basic Problems**.

Integration by Substitution

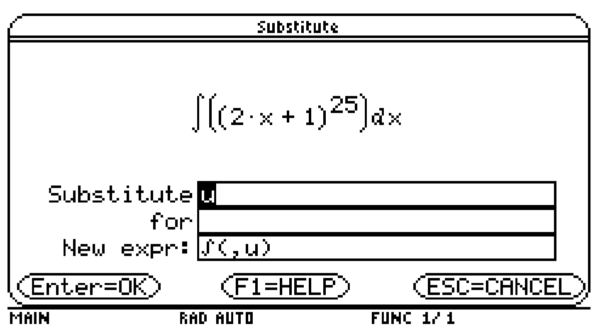
Integration by substitution is one of the most important methods of integration for beginning calculus students to master. See Student Reference Sheet 2, **Key Transformations,** for a table of the most important transformations. The substitution method is based on the chain rule. Refer to your textbook for more details about the chain rule.

Learning to use integration by substitution greatly increases your ability to obtain antiderivatives. As a first example, consider $\int (2x+1)^{25}\,dx$. Enter this problem into your handheld in the same manner as shown on the prior page. When the problem is entered, perform the following steps on your handheld.

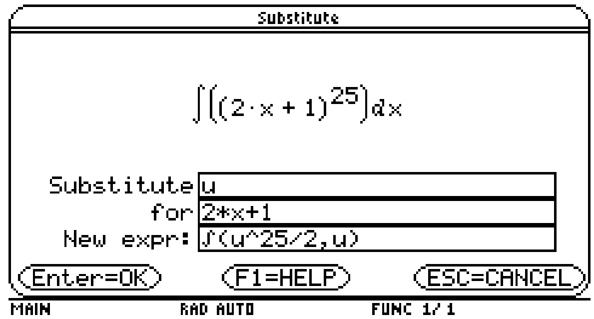
1. Press [F4] to list possible actions.



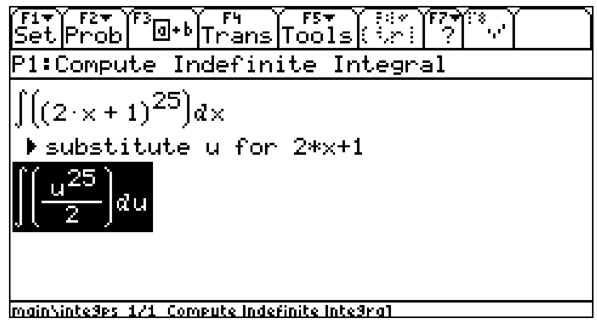
2. Press **1** to select integration by substitution.



3. Use pencil and paper to decide how to make the substitution, and then press [F1] to have SMG fill in the key information. Make sure that the substitution is the one you want. Change it if it is not.



4. Press [ENTER] until the dialog box disappears.



More steps (without these details for the substitution process) are in Student Work Sheet 2, **Integration by Substitution**. You will find nine more problems to enter and try with SMG. Remember that the goal is to learn how to perform these calculations without SMG's help. Learn how to see compositions in the integrands and look for derivatives of the inside functions.

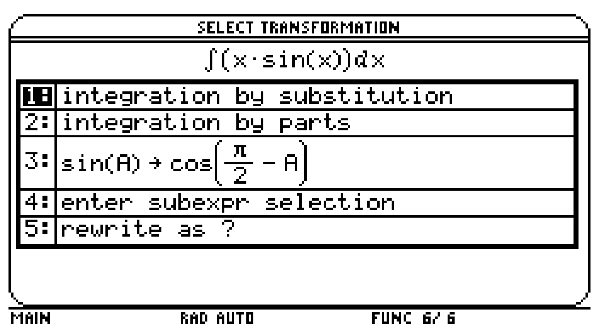
Integration by Parts

Another very important method of indefinite integration is integration by parts. It is based on the product rule for differentiation.

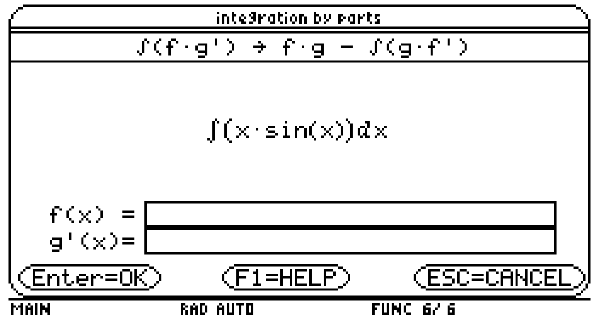
<u>Example</u>

Enter the problem, $x \cdot \sin(x)dx$, and follow along with the steps that follow.

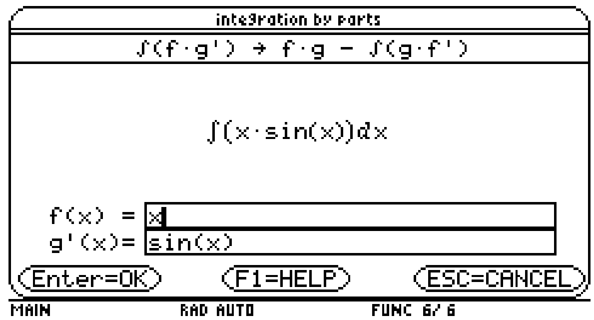
1. Press [F4] to list possible actions.



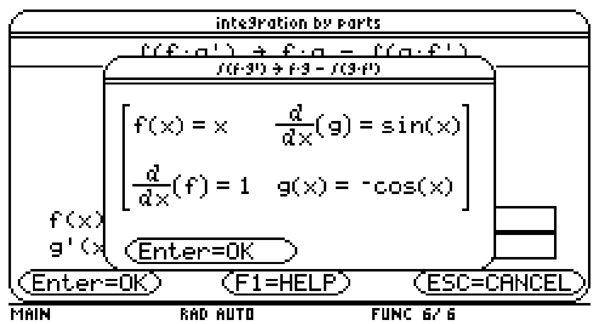
2. Press **2** to select integration by parts.



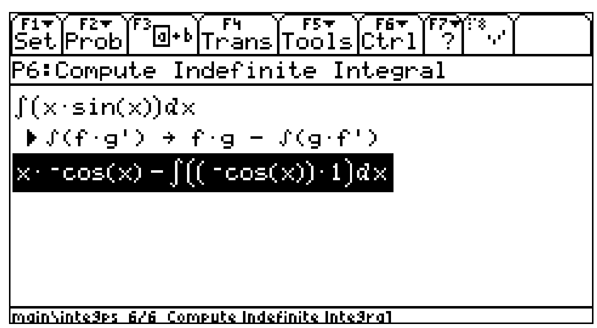
3. Decide how to choose f(x) and g'(x), and then press [F1] to have SMG fill in the key information. Make sure that the choice is the one you want. Change it if it is not.



4. Press [ENTER] until a new dialog box is displayed.



5. Press [ENTER].



For additional steps, refer to Student Work Sheet 3, **Integration by Parts**. This also contains nine more problems for you to enter and solve using SMG.

Partial Fraction Decomposition

The need for this method arises naturally when solving differential equations. This activity does not include a special example here because applying partial fractions does not cause SMG to show any special windows or request that you fill in any additional information. To help you learn to use partial fraction decomposition, Student Work Sheet 4, **Integration by Partial Fractions,** contains five problems for you to enter and solve. Several of these are provided in steps format while the remaining problems contain only hints for solving them.

Developing Successful Strategies

With SMG, you can accurately work lots of problems. Try this experience to help you develop successful strategies more quickly. This section provides some useful thoughts about developing strategies.

Substitution

For most problems in textbooks, find a factor of the integrand that is a composition (*u*). The degenerate case corresponds to (*u*) = *u*. Let $u = g(x)$ denote the inside function.

<u>Common Case</u>

The $\frac{\text{integrand}}{f(u) \cdot g'(x)}$ is a constant. Make the substitution, adjusting for the constant. Automatically apply the scalar product formula. To see how this works, try it on several examples.

<u>Unusual Case</u>

The $\frac{\text{integrand}}{f(u) \cdot g'(x)}$ is not a constant. Try to solve $u = g(x)$ for *x* (textbook problems usually require solving for *x*^2 and are very easy) and substitute into $h(x)$. Make the substitution and apply the scalar product formula. After working all the examples in Student Work Sheet 2, **Integration by Substitution**, look through your solutions to find any that match this case.

<u>Other Rare Cases</u>

Look through the examples in Student Work Sheet 2, **Integration by Substitution,** for other rare cases.

Integration by Parts

For most problems found in textbooks, try the following strategy. If the integrand has several factors, choose *u* to correspond to the first type on the list (logarithmic, inverse trigonometric, algebraic (usually a power function), trigonometric, or exponential).

An exception is integrands of the form U^n, where *n* is a positive integer > 1, and U is a basic trigonometric function such as sin(*x*) or cos(*x*). These lead to reduction formulas. In this case, peel off one factor as *dv* and let *u* = the rest.

Check these strategies against your solutions to the problems in Student Work Sheet 3, **Integration by Parts**.

Construct a General Plan

How do you decide whether to try substitution, integration by parts, partial fractions, or a basic transformation? Write an algorithm for choosing what to try first. This could be challenging, but trying to do this should help you improve your skills (or show you what you need to work on).

Once your plan is devised, test it using the problems contained in Student Work Sheet 5, **You Pick the Method(s)**.

Student Reference Sheet 1
Basic Indefinite Integrals

Name: ______________________________

Date: ______________________________

The following table contains a small sample of the formulas available.

Basic Transformations	**Examples/Comments**
$\int x \cdot dx \rightarrow \frac{1}{2} \cdot x^2 + C$	Transformations produce general indefinite integrals. The general constant is not shown below.
integral of constant	$\int k dx \rightarrow k \cdot x + C$
$\int (x^r) dx \rightarrow \frac{1}{r+1} x^{r+1}$	$\int (x^2) dx \rightarrow \frac{1}{2+1} x^{2+1}$
integral of polynomial	$\int \left(x^2 + 4x + 1\right) dx \rightarrow \frac{1}{2+1} x^{2+1} + 4 \cdot \frac{1}{2} x^2 + 1 \cdot x + C$
$\int \frac{1}{x-a} dx \rightarrow \ln(\|x-a\|)$	$\int \frac{1}{x} dx \rightarrow \ln(\|x\|)$
$\int a^x dx \rightarrow \frac{a^x}{\ln(a)}$	$\int e^x dx \rightarrow e^x$
$\int \frac{1}{a+x^2} dx \rightarrow \frac{1}{\sqrt{a}} \cdot \tan^{-1}\left(\frac{x}{\sqrt{a}}\right)$	$\int \frac{1}{1+x^2} dx \rightarrow \tan^{-1}(x)$
$\int \frac{1}{\sqrt{a-x^2}} dx \rightarrow \sin^{-1}\left(\frac{x}{\sqrt{a}}\right)$	$\int \frac{1}{\sqrt{1-x^2}} dx \rightarrow \sin^{-1}(x)$

Student Reference Sheet 2
Key Transformations

Name: ______________________

Date: ______________________

Transformations	Examples/Comments
integral of sum or diff	$\int x+\sin(x)dx$
$\int(c\cdot f(x))dx \to c\cdot\int f(x)dx$	$\int 3\cdot\ln(x)dx$
integration by substitution $\int f(g(x))\cdot g'(x)dx \to \int f(u)du$	$\int 2x\cdot e^{x^2}dx$
integration by parts $\int(f\cdot g') \to f\cdot g - \int(g\cdot f')$	$\int x\cdot e^{x}dx$
apply partial fractions	$\int\frac{1}{(x-2)\cdot(x-1)}dx = \int(\frac{1}{x-2}-\frac{1}{x-1})dx$
basic integrals	$\int xdx + \int\cos(x)dx$ Applies all applicable basic integral formulas at one time.

Student Work Sheet 1

Basic Problems

Name: ____________________

Date: ____________________

Use these problems to check your mastery of the basic formuals. Compute each of the indefinite integals below using SMG unless indicated otherwise.

1. $\int x^4 dx$ [Hint: Use the power rule]

2. $\int 3 \cdot x^4 dx$

3. $\int 3 \cdot x^4 + 5dx$

4. $\int 3 \cdot x^4 + x^2 + 5dx$ [Instructor's note: Indicate if the students can use *integral of polynomial.*]

5. $\int \cos(x) dx$

6. $\int \tan(x) dx$

7. $\int 3^x dx$

8. $\int \frac{1}{x^4} dx$

9. $\int \ln(x) dx$

10. $\int \log_2(x) dx$ [Hint: Type $\log_2(x)$ as ln(x)/ln(2)]

Student Work Sheet 2

Integration by Substitution

Name: ______________________________

Date: ______________________________

You should work several problems manually before using SMG to work the problems below.

1. $\int (2x+1)^{25}\,dx$

   Compute using the following SMG steps:

   a. Substitute *u* for *x*+1 [Hint: Refer to the activity section, *Integration by Substitution*, for details.]

   b. $\int c \cdot f dx \rightarrow c \cdot \int (f)$ (scalar product)

   c. $\int (x^r) dx \rightarrow \frac{1}{r+1} x^{r+1}$ (power rule)

   d. Back substitute 2*x*+1 for *u.*

2. $\int \frac{e^x}{1+e^x}\,dx$

3. $\int 2x \cdot (x^2+5)^7\,dx$

4. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

5. $\int \tan(x)dx$. Before using SMG to solve this problem, use the steps below to help you compute the indefinite integral by-hand. Check your steps with SMG.

   a. tan(*A*) -> sin(*A*)/cos(*A*)

   b. Substitute *u* for cos(*x*)

   c. $\int c \cdot f dx \rightarrow c \cdot \int (f)$

   d. $\int \frac{1}{x}\,dx \rightarrow \ln(|x|)$

   e. Back substitute cos(*x*) for *u*, then clean up the answer.

6. $\int \cos(2x)dx$

7. $\int \sin(x) \cdot \cos(x)dx$

8. $\int \frac{x+1}{x^2+1}dx$

9. $\int \sqrt{1+x^2} \cdot x^5 dx$ [Hint: You are on you own. F1 Help does not help in solving this problem. Try the integral substitution *u* = 1+*x*^2 . Then *x*^4 would be replaced by (1-*u*)^2.]

10. $\int \cos(x)^3 dx$ [Hint: Rewrite the integrand as cos(*x*)*cos(*x*)^2.]

Student Work Sheet 3
Integration by Parts

Name: ______________________________

Date: ______________________________

Work several problems by-hand before using SMG to help you with the problems below.

1. $\int x \cdot \sin(x)dx$. Compute using the following SMG steps:

   a. $\int (f \cdot g') \rightarrow f \cdot g - \int (g \cdot f')$

   b. $\int (c \cdot f(x))dx \rightarrow c \cdot \int f(x)dx$

   c. $\int \cos(x)dx \rightarrow \sin(x)$

   d. Simplify by pressing [ENTER].

2. $\int \ln(x)dx$ [Hint: ln(x) = 1*ln(x)]

3. $\int x^2 \cdot e^x dx$ [Hint: Apply integration by parts twice.]

4. $\int \tan^{-1}(x)dx$

5. $\int \sin^n(x)dx$, where *n* integer and *n* ≥2.

6. $\int x \cdot \cos(3x)dx$

7. $\int \sin(\ln(u))du$

8. $\int \sqrt{x} \cdot \ln(3x)dx$

9. $\int x \cdot \tan^{-1}(x)dx$

10. $\int e^x \cdot \cos(x)dx$ [Apply integration by parts twice, then finish solving manually by setting the original integral equal to the final expression and then solving for the integral.]

Student Work Sheet 4
Integration by Partial Fractions

Name: ______________________

Date: ______________________

Compute each of the indefinite integals below using SMG unless other instructions are provided. Simplify integrands that are ratios of polynomials

1. $\int \frac{1}{(x-1)\cdot(x+1)}dx$

   Compute the answer with SMG using the following steps:

   a. Apply partial fractions.

   b. Integral of sum or diff.

   c. $\int (c\cdot f(x))dx \rightarrow c\cdot \int f(x)dx$

   d. $\int \frac{1}{x}dx \rightarrow \ln(|x|)$

2. $\int \frac{1}{x^2-1}dx$

3. $\int \frac{5x-4}{2x^2+x-1}dx$

4. $\int \frac{1}{(x-1)\cdot(x^2+4)}dx$

5. $\int \frac{x^2}{x+1}dx$

   Compute the answer with SMG using the following steps:

   a. Apply partial fractions (long division if doing this by-hand).

   b. Integral of sum or diff.

   You should have the sum/difference of three basic integrals. You can finish all at once by choosing basic integrals or in three steps by choosing individual transformations one at a time. Remember to simplify in order to clean-up the final result.

Student Work Sheet 5

You Pick the Method(s)

Name: ____________________

Date: ____________________

Compute each of the indefinite integals below using SMG unless other instructions are provided.

1. $\int x \cdot \sin(2x)dx$ [Hint: You can use substitution followed by integration by parts. Can you start with integration by parts instead?]

   a. Substitute *u* for 2*x*

   b. Parts with *f*(*u*) = *x* and *g*′(*u*) = sin(*u*)

2. $\int \frac{u}{a} \cdot \cos(u)du$

3. $\int \frac{1}{x - \frac{a^2}{b^2}} dx$

4. $\int \cos(\ln(u))dx$ [Hint: You probably do not need SMG for this one.]

5. $\int \frac{x+1}{2x^2+3x+1} dx$

6. $\int (x^2+1)^6 \cdot x dx$

7. $\int x^3 \cdot \sqrt{9-x^2}\, dx$ [Hint: You must determine your own substitution.]

8. $\int \sin^3(u)du$

9. $\int \sin(x)\cdot\cos(\cos(x))dx$

10. $\int \frac{dt}{t^2+6t+8}$