Approximating the Area Under a Curve

About the Lesson

In this activity, students will explore approximating the area under a curve using left endpoint, right endpoint, and midpoint Riemann sums. As a result, students will:

- Develop an understanding of summation notation for adding these rectangles.
- Explore the trapezoidal sum approximation for area and compare these various approximations methods.

Vocabulary

- summation notation
- left Riemann sum
- right Riemann sum
- midpoint Riemann sum
- trapezoidal sum

Teacher Preparation and Notes

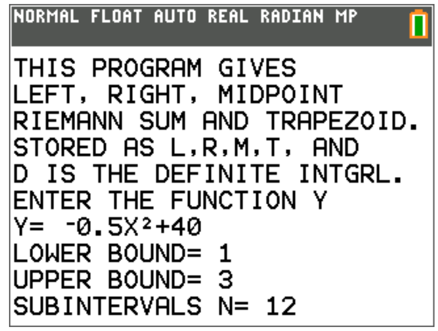
- This activity will utilize the **AREAPPRX** program that must be installed on the students' calculators prior to beginning the activity.

Activity Materials

- Compatible TI Technologies:

  TI-84 Plus*
  TI-84 Plus Silver Edition*
  TI-84 Plus C Silver Edition
  TI-84 Plus CE

** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials
- Any required calculator files can be distributed to students via handheld-to-handheld transfer.

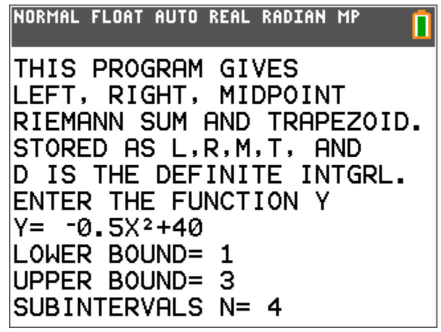
Lesson Files:

- Approximating_the_Area_Under_a_Curve_Student.pdf
- Approximating_the_Area_Under_a_Curve_Student.doc
- AREAPPRX.8xp

Tech Tip: Before beginning the activity, the program AREAPPRX.8xp needs to be transferred to the students' calculators via handheld-to-handheld transfer or transferred from the computer to the calculator via TI-Connect.

Problem 1 – Graphical Riemann Sums

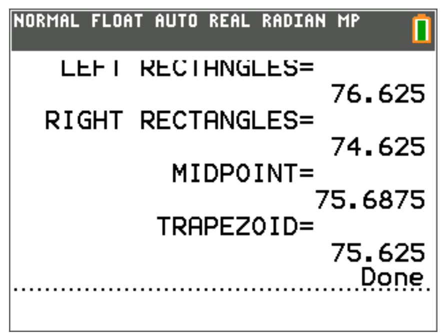
Students will be presented with the function, $f(x) = -0.5x^2 + 40$, and be asked to calculate three different approximations for the area under its curve on the interval $x = 1$ to $x = 3$.



They will perform these calculations by first executing the program, **AREAPPRX** by pressing [prgm] and arrowing down until they reach **AREAPPRX**. They will then press [enter] twice.

The program will prompt the student for several pieces of information as shown to the right. The students will be instructed to use 4 subintervals for their first approximation sums. They will record the results of the left rectangle, right rectangle, and midpoint sum and repeat this process for 12 subintervals.

Example 1: Record the following three types of approximations below.



Using 4 rectangles:

Left Riemann sum = _____________

Right Riemann sum = _____________

Midpoint Riemann sum = _____________

<u>**Answers**</u>: 76.625; 74.625; 75.6875

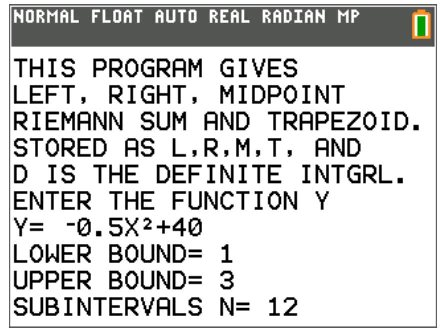
Using 12 rectangles:

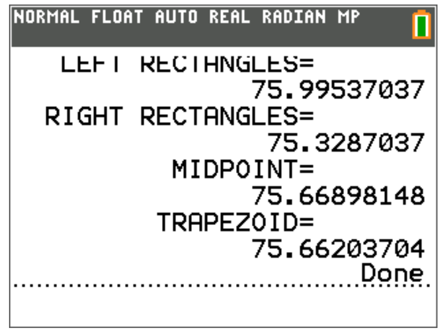
Left Riemann sum ≈ ____________

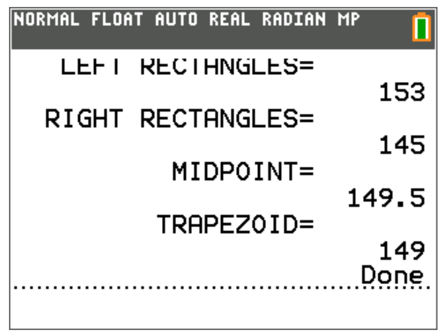
Right Riemann sum ≈ ____________

Midpoint Riemann sum ≈ ____________

Answers: 75.9954; 75.3287; 75.6690







The student will be asked to calculate the value of the definite integral, $\int_1^3 (-0.5x^2+40)dx$ using **fnInt** by pressing [math], choosing option **9:**, and entering the fields with the appropriate information.

Example 2:

$$\int_1^3 (-0.5x^2+40)dx =$$

Answer: $75.\overline{6}$

Students will be asked to compare this result with the four approximations of the area found earlier.

Teacher Tip: This would make a good time to start a discussion with students about how the shape of the graph affects how the exact area compares with each of the four approximation methods. If during the discussion, words like "increasing," decreasing" and "concavity" aren't brought up by the students, you may prompt them by asking about the whether those attributes have an effect upon the comparison of those areas.

1. Letting $y = -0.5x^2 + 40$ again, run the **AREAPPRX** program from $x = 0$ to $x = 4$ and use 4 rectangles. How do the left, midpoint, and right Riemann sums compare? Explain why.

Answer: The right endpoint approximation is the smallest value whereas the left endpoint approximation is the largest. The midpoint approximation falls in between the two.

2. Describe what happens to the left, midpoint, and right Riemann sums as you increase the number of subintervals, *n*.

Answer: The Riemann sums converge (to the definite integral) as $n \to \infty$. With thinner widths, the rectangles approach the true area.

3. Is the midpoint Riemann sum an over or under approximation if the graph is:

a. Increasing and concave down? ____ over _____ under

b. Increasing and concave up? ____ over _____ under

c. Decreasing and concave down? ____ over _____ under

d. Decreasing and concave up? ____ over _____ under

After graphically exploring (especially with a small number of subintervals), explain why.

Answers: (a) over; (b) under; (c) over; (d) under; When the function is decreasing and concave down, the function curves more steeply for the second half of the rectangle than the first. Because there is more rectangle above than below the graph, it over estimates the area.

Having students use the **AREAPPRX** program allows for many questions to be asked and for observations to be made. This section can be extended by asking about other functions. Encourage students to graph other functions that have different concavity or slope to complete Question 3 above.

Problem 2 – Summation Notation

In Problem 2, students examine the summation notation. By reading and answering questions on the student worksheet, students will better understand the sigma notation.

To make entering the summation easier, encourage students to define the function before entering the sum.

Examine the function $Y_1(x) = -0.5x^2 + 40$.

4. The thickness of each rectangle is $\Delta x = h = \frac{b-a}{n}$. If $a = 1$, $b = 6$, and $n = 5$. What is Δx?

Answer: $\Delta x = h = 1$

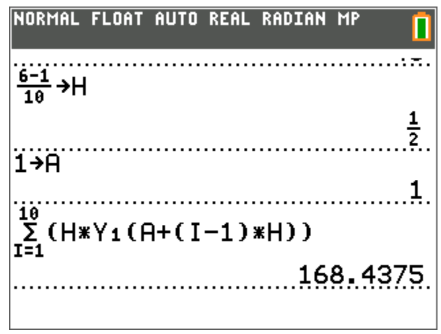
5. Expand $\sum_{i=1}^{5}(1 \cdot Y_1(a+(i-1)\cdot 1)$ by writing the sum of the five terms and substituting i = 1, 2, 3, 4, and 5.

Answer: Y1(a) + Y1(a + 1) + Y1(a + 2) + Y1(a + 3) + Y1(a + 4)

6. Explain why this is the summation notation for LEFT Riemann sums and not the RIGHT.

Sample Answer: The right Riemann sums from a to b with 5 subintervals, looks like Y1(a + 1) + Y1(a + 2) + Y1(a + 3) + Y1(a + 4) + Y1(a + 5). a + 5 = b because Δx = 1. The i –1 component of the above summation is usually indicative of a left Riemann sum.

7. Let $y(x) = -0.5x^2 + 40$, a = 1, and b = 6. Write the sigma notation and use the HOME ([2nd][mode][quit]) screen to evaluate the left Riemann sum for 10, 20, 50, and 100 subintervals.



a. n = 10

b. n = 20

c. n = 50

d. n = 100

Answers: $\sum_{i=1}^{n}\left(h\cdot Y_1\left(a+\left(i-1\right)h\right)\right)\Big| h=\frac{6-1}{n}$; (a) 168.4375; (b) 166.328125; (c) 165.0375; (d) 164.603125

Extension – Area Programs

Use the Area Approximation program **AREAPPROX** to answer the following questions.

8. Let $y(x) = x^2$, a = 1, and b = 6. Write the results for midpoint and trapezoid area approximations when

a. n = 10

b. n = 50

c. n = 100

Answers: (a) midpoint sum = 71.5625; trapezoidal sum = 71.875; (b) midpoint sum = 71.6625; trapezoidal sum = 71.675; (c) midpoint sum = 71.6656; trapezoidal sum = 71.6688

9. Compare the above midpoint and trapezoid values with the actual area.

Answer: For $y = x^2$, an increasing concave up function in this domain, the midpoint is approaching $71\frac{2}{3}$ from the left and the trapezoid method is approaching the value of the definite integral from the right. Midpoint gives a better approximation.

