EXPLORATIONS

Chapter 2

Recovering a Function from its Derivative: A Numerical Approach

In Chapter 1, you studied the problem of recovering a function from its derivative. You learned how to use the TI-86 to obtain a graph of the function when you knew the derivative of the function. The graph is a visual solution to a differential equation.

In this chapter, you will learn how to find specific ordered pairs for a function when you know its derivative and a point on the function. The ordered pairs are a *numerical* solution to a differential equation. This chapter provides several examples that illustrate how to find numerical solutions.

Example 1: A Table of Values for a Differential Equation

If $\frac{dy}{dx} = 3x^2$ and $y = 1$ when $x = 0$, what is the value of y when $x = 1$?

What is the value of y when $x = 2$? When $x = 3$?

Solution

Use the TI-86 to find the graph of y, then use the graph trace feature to find the y values.

1. Use the format screen (**FORMT**) to turn the slope field off. (Figure 2.1)

Figure 2.1

2. Using the axes editor, select **t** for the x-axis and **Q** for the y-axis. (Remember that the TI-86 uses **t** in place of **x** and **Q** in place of **y**.) (Figure 2.2)

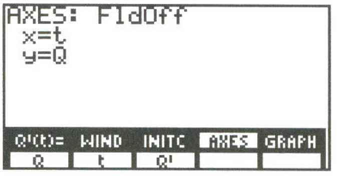

Figure 2.2

3. Using the differential equation editor (**Q'(t)=**), enter the differential equation and select a thick graphing style. (Figure 2.3)

Figure 2.3

4. Using the initial conditions editor, enter the initial conditions. The initial value of **x** is assigned to **tMin** and the initial value of **y** is placed in **QI1**. (Figure 2.4)

Figure 2.4

5. Select the window editor and enter **tMin** = 0, **tMax** = 4 and **tStep** = .1. Set the viewing window at [-4, 4] x [-15, 15] by entering ⁻4 for **xMin**, 4 for **xMax**, ⁻15 for **yMin**, and 15 for **yMax**. (Figures 2.5 and 2.6)

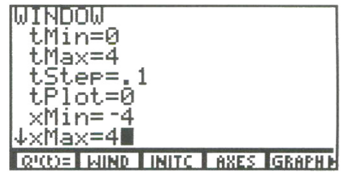

Figure 2.5

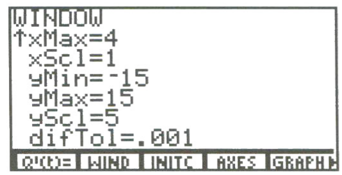

Figure 2.6

6. Press [F5] (**GRAPH**) to display the solution. (Figure 2.7)

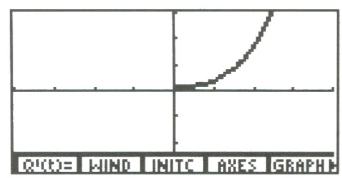

Figure 2.7

7. You can now use the **TRACE** feature to find the value of y when x is 1, 2, or 3. To access this feature from the GRAPH menu, press [MORE] [F4].

   You should see the trace cursor blinking at the center of the screen and the coordinates of the cursor at the bottom of the screen. You can move the trace cursor with the left and right cursor movement keys. (Figure 2.8)

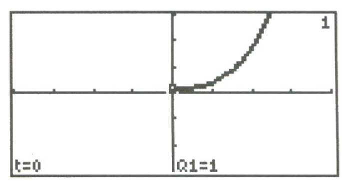

Figure 2.8

8. Press [▶] to move the trace cursor to the right until **t** = 1. (Figure 2.9)

   The coordinates at the bottom of the screen show that **y** = 2 when **x** = 1. (Remember that Q1 = **y** and t = **x**.)

   Instead of using the cursor movement keys to move the trace cursor, you can jump directly to the desired *t*-coordinate by typing in the value of **t** directly. You will use this method to find the *y*-coordinates for **x** = 2 and **x** = 3.

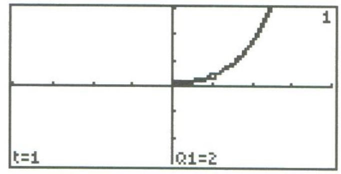

Figure 2.9

9. While the trace cursor is still blinking, press [2] [ENTER]. (Figures 2.10 and 2.11)

   There will be a short delay before the cursor moves to the point where **t** = 2. This is because the TI-86 begins at the initial point and makes successive calculations to reach the desired point. The result of the **TRACE** feature shows that when **x** = 2, **y** = 9.

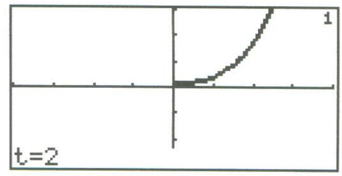

Figure 2.10

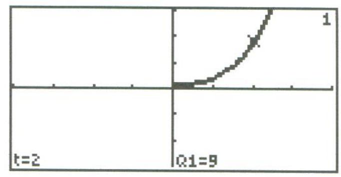

Figure 2.11

10. Press [3] [ENTER] to move the cursor to **t = 3**. (Figure 2.12)

    When **x** = 3, **y**= 28. Notice that the coordinates appear at the bottom of the screen in Figure 2.12, even though the trace cursor has moved above the viewing window.

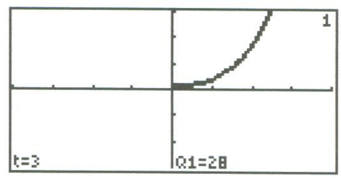

Figure 2.12

You can summarize the ordered pairs in a table using the table feature of the TI-86.

1. Press [TABLE] to display the table menu. (Figure 2.13)

2. To prepare the table, use the table setup editor (**TBLST**). To display this screen, press [F2]. (Figure 2.14)

3. Enter the values shown in Figure 2.14.

   TblStart is the initial value of **t** in the table. Δ**Tbl** is the amount by which **t** increases from one row to the next in the table. When **Auto** is selected, the TI-86 will automatically generate values in the table. If **Auto** is not selected on the last row, use the cursor movement keys to highlight **Auto** and then press [ENTER]. Press [F1] to create the table. Before you see the table, there will be a delay while the TI-86 calculates the table entries.

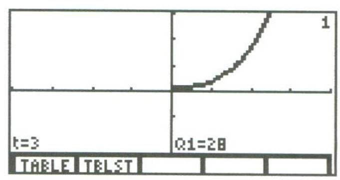

Figure 2.13

Figure 2.14

Figure 2.15 shows the table, which includes the initial conditions and all the ordered pairs you found with the trace feature as well as an additional ordered pair. It also shows an error for **t** = 5, because the maximum value of **t** in the window editor was 4. The table will not find values beyond **tMax**.

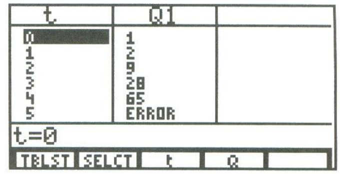

Figure 2.15

Since all the ordered pairs in the table lie on the graph of

$$y = f(x),$$

we say that the table is a *numerical* solution to the initial value problem:

$$\frac{dy}{dx} = 3x^2, \; y = 1 \text{ when } x = 0.$$

Compare the results in the table to the values given by the analytic solution to this differential equation,

$$y = x^3 + 1.$$

Example 2: Exponential Growth

Solve the initial value problem

$$\frac{dy}{dx} = 3x^2, \; \mathrm{x} = 0, \; \mathrm{y} = 1$$

graphically and numerically.

Solution

1. The axes editor should show that **x= t** and **y = Q** and that the Field is turned off (**FldOff**). (Figure 2.16) If this is not the case, make the appropriate changes in the axes editor and format screen.

Figure 2.16

2. Since **x = t** and **y = Q**, you will enter the differential equation

$$\frac{dy}{dx} = y$$

as **Q'1** = Q1. (Figure 2.17)

Figure 2.17

3. Enter the initial conditions as **tMin** = 0 and **QI1** = 1. (Figure 2.18)

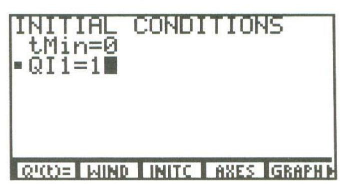

Figure 2.18

4. Enter the viewing window values shown in Figures 2.19 and 2.20.

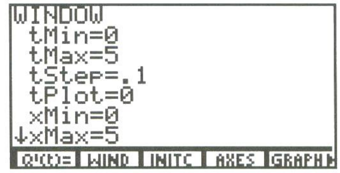

Figure 2.19

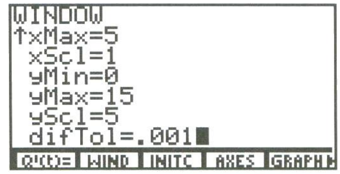

Figure 2.20

5. To view the graphical solution, press [F5] (**GRAPH**) (Figure 2.21). To remove the menu from the bottom of the screen, press [CLEAR].

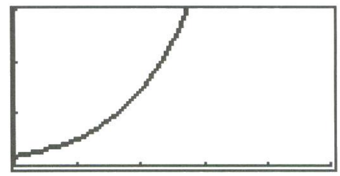

Figure 2.21

The analytic solution to this differential equation is

$$y = e^x .$$

To compare the analytic solution to the graphical solution, use the **DrawF** command, which is found on the GRAPH DRAW menu. (Figures 2.22 and 2.23) The solutions appear to be the same since there is only one graph.

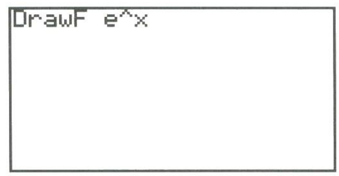

Figure 2.22

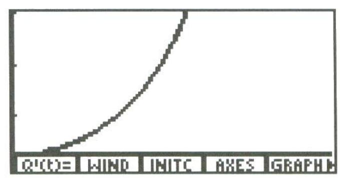

Figure 2.23

You can find the numerical solution with the table setup editor and table feature.

1. To select the table setup editor, press [TABLE] [F2] (**TBLST**).

2. Enter the values shown in Figure 2.24.

3. Press [F1] to see the table. (Figure 2.25)

Figure 2.24

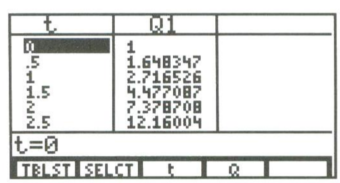

Figure 2.25

Notice the entry for **t** = 1. This value should be **e**, which is about 2.71828, but the table entry is 2.716526. This shows us the solution obtained using **DifEq** mode is an approximation that differs slightly from the analytic solution.

If you select the **Ask** feature in the table setup editor, you can enter various values of **t** and see the corresponding values of **y.** Return to the table setup editor by pressing [F1] (**TBLST**). Then use the cursor movement keys to highlight **Ask** and press [ENTER]. (Figure 2.26)

Figure 2.26

Press [F1] to see the table. It should be blank. Now you can enter any value of **t** between **tMin** and **tMax**. Type 2.25, and then press [ENTER]. (Figure 2.27)

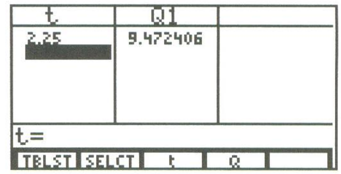

Figure 2.27

Try entering other values for **t.**

Example 3: Compound Interest

The table at the right shows how an initial investment of $1000 increases each year over a five year period. If the investment continues to grow at a similar pace, what should the amount be after ten years?

Year	Amount	Increase
0	1000.00	-------
1	1099.66	99.66
2	1209.25	109.59
3	1329.76	120.51
4	1462.28	132.52
5	1608.01	145.73

Table 2.1

Solution

The amount increases each year by about 10%. The increase per year is a rate of change that is similar to a derivative. If t equals time in years and y equals the amount of the investment after t years, then

$$\frac{dy}{dt}$$

is an approximation for the yearly increase.

The statement *the amount increases each year by 10%* is similar to the differential equation

$$\frac{dy}{dt} = 0.10y\ .$$

You can use this differential equation to generate a table that approximates Table 2.1.

1. To create the table, enter the differential equation

$$\frac{dy}{dt} = 0.10y$$

in the TI-86 as **Q'1** = 0.10 Q1. (Figure 2.28)

Figure 2.28

2. After you enter the equation, enter the initial conditions. (Figure 2.29)

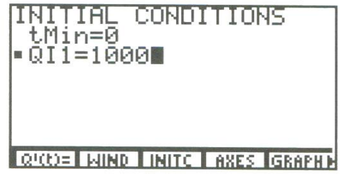

Figure 2.29

3. Press [F2] (**WIND**) to display the window editor. Since you want to make a table for years 0 through 10, enter 0 for **tMin** and 10 for **tMax**. The other values in the window editor will not affect the table. (Figure 2.30)

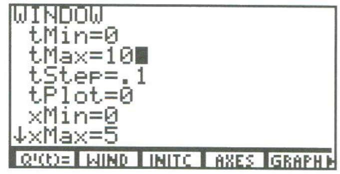

Figure 2.30

4. The table results should show only two decimal places, so you must modify the mode settings.

   Press [2nd] [MODE] and then use the cursor movement keys to highlight **2** on the second line of the mode settings screen. Press [ENTER]. This fixes the number of decimal places at 2. (Figure 2.31)

Figure 2.31

5. Use the table setup editor to enter an initial value of 0 for **t** and 1 for Δ**Tbl**. Select **Auto** on the last line of the menu. (Figure 2.32)

Figure 2.32

6. Generate the table [F1] (**TABLE**). The resulting table approximates Table 2.1 but is not quite the same. This is because

$$\frac{dy}{dt}$$

is only an approximation for the increase per year in the amount. (Figure 2.33)

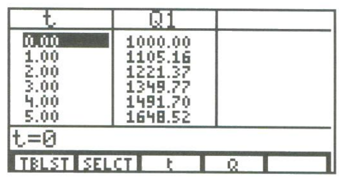

Figure 2.33

7. You can obtain a table that more closely matches Table 2.1 by adjusting the coefficient of **Q1** in the differential equation. By trial and error, you can discover that a coefficient of 0.095 gives closer results. (Figures 2.34 and 2.35)

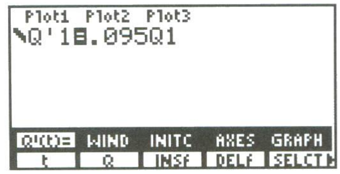

Figure 2.34

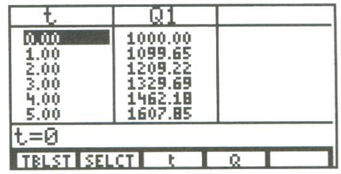

Figure 2.35

8. You can scroll down the table with the cursor movement keys. Press [▼] until you see the row with $t = 10$. (Figure 2.36)

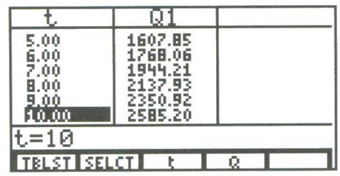

Figure 2.36

You can now estimate the amount in ten years to be about $2,585. The differential equation

$$\frac{dy}{dt} = .095y \quad (Q'1 = .095Q1)$$

corresponds to a continuous growth rate of 9.5%. This is very close to a discrete annual growth rate of 10%.

Exercises

1. Make a graph and a table to show the solution to the differential equation:

$$\frac{dy}{dx} = -2xy^2, \quad x = -2, \quad y = 0.2$$

2. Solve the differential equation graphically and numerically (that is, make a graph and a table of the solution). Change the mode to **Float** before making the table.

$$\frac{dy}{dx} = x + y, \; x = {}^-1.3, \; y = 0.6.$$

3. Doubling time for an investment: An investment is known to grow according to the differential equation

$$\frac{dy}{dt} = .0875y.$$

How long will it take for an investment of $1000 to double?

Hint: *Make a table and use the Ask feature to guess and correct.*

4. Use the **TRACE** feature to find the value of the investment in Exercise 3 after 6.5 years.

5. Half-life. Carbon-14 is known to decay according to the differential equation

$$\frac{dy}{dt} = -0.00012y.$$

How old is a sample in which 10% of the radioactive material originally present has decayed? Assume there were 4 grams of the radioactive material originally present.

Hint: *Make a table and use the Ask feature to guess and correct.*

6. Rework Exercise 5 using the assumption that there were 12 grams of the radioactive material originally present. How does this affect the calculated age of the sample?

7. Rework Exercise 5 using the assumption that there were 23 grams of the radioactive material originally present. How does this affect the calculated age of the sample? Make a conjecture about the effect of the original amount of radioactive material on the calculated age of the sample. Prove your conjecture analytically.