The Simson Line

Construct and Investigate:

1. Using the **Triangle** tool, draw $\triangle DEF$ in the center of the Voyage™ 200 screen. Construct its circumcircle. The circumcircle's center O is the intersection of the perpendicular bisectors of the sides of $\triangle DEF$. Put point P anywhere on the circumcircle. Hide the perpendicular bisectors. Extend the sides of $\triangle DEF$ by constructing lines $\overleftrightarrow{DE}$, $\overleftrightarrow{EF}$, and $\overleftrightarrow{FD}$. Construct perpendicular lines through P to each of the lines $\overleftrightarrow{DE}$, $\overleftrightarrow{EF}$, and $\overleftrightarrow{FD}$, and label the points of intersection R, S, and T (Figure 1).

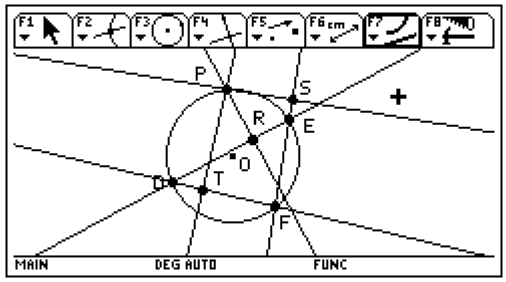

Figure 1

   Note: Points R, S, and T must be the intersections of the perpendicular lines with the extension lines and not with the sides of $\triangle DEF$.

   Hide the three lines through the vertices and the three perpendicular lines, leaving only $\triangle DEF$, its circumcircle, and the points P, R, S, and T (Figure 2). Drag point P on the circumcircle. Use the **Check Properties** tool to make a conjecture about the relationship between points R, S, and T. Construct the line that contains R, S, and T. This line, as shown in Figure 2, is commonly called the **Simson line**.

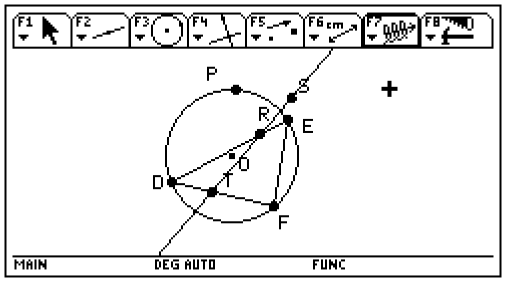

Figure 2

2. Drag the vertices of $\triangle DEF$ and point P along the circumcircle, and note the behavior of the Simson line. Describe the relationship of the Simson line and points R, S, and T to $\triangle DEF$.
3. Make a macro construction named **Simson Line**. The **Initial Objects** should be $\triangle DEF$, its circumcircle, and point P. The **Final Object** should be the Simson line. Hide points R, S, and T for the next explorations. Use the **Simson Line** macro to explore which points on the circumcircle have a Simson line that passes through each of the sides of $\triangle DEF$.
4. For any three points on the circumcircle of $\triangle DEF$, what relationship exists between the triangle formed by these points and the one formed by the intersection of their corresponding Simson lines?

Explore:

1. Use the **Animation** tool to animate point P on the circumcircle, and make a conjecture about the locus of the Simson line as P moves around the circumcircle of $\triangle DEF$. Use the **Locus** tool to construct the envelope of Simson lines as point P moves around the circumcircle of $\triangle DEF$. Drag the vertices of the original triangle around the screen. Refine your conjectures about the relationship(s) that you observe.
2. What is the relationship between the Simson lines of two diametrically opposite points on the circumcircle? What is the locus of the intersection point of the two Simson lines for diametrically opposite points on the circumcircle of a triangle. What relationship exists between this locus and the one you found in part 1? What is the relationship between this locus and the circumcircle of the triangle? Explore other properties that appear to be true about this locus of points.

Teacher's Guide: The Simson Line

Construct and Investigate:

1. Students need to work carefully through the instructions in this activity. Hiding lines that are no longer needed helps keep the screen from getting too cluttered. When extending the sides of $\triangle DEF$, make sure that students use the vertices of the triangle as the points that define the line. This ensures that the lines really are extensions of the sides of $\triangle DEF$. Students can use the **Thick** tool to make the important points more noticeable. It may come as a surprise that points ***R***, ***S***, and ***T*** are collinear (Figure 3).

   This line is most often called the **Simson line** for Robert Simson (1687–1768), although there is no evidence that he knew about its existence. Other sources give credit for the discovery of the line to William Wallace in 1797 and refer to it as the **Wallace line**.

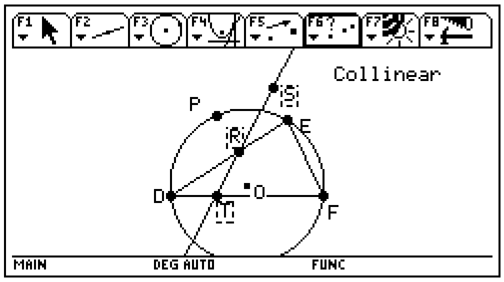

Figure 3

2. If point ***P*** is on a vertex of $\triangle DEF$, then two of the three points, ***R***, ***S***, or ***T***, are also located at this vertex and the Simson line passes through point ***P*** (Figures 4 and 5). Because ***P*** is on a vertex, the perpendicular lines from ***P*** to the extension of the two sides meeting at this vertex are zero in length. This is true because the two side extensions pass through the vertex, as well as through point ***P***.

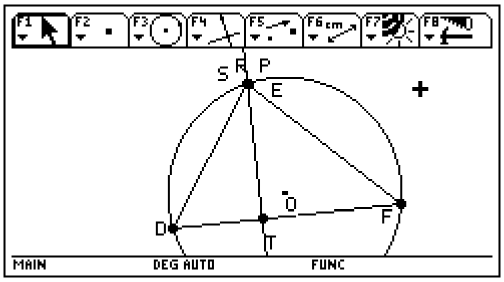

Figure 4

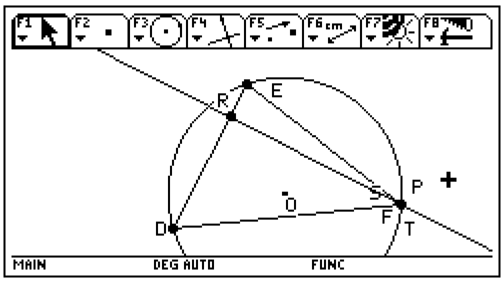

Figure 5

3. If point ***W*** is on the circumcircle of $\triangle DEF$ at a point that is directly opposite vertex ***F***, then the Simson line defined for point ***W*** contains side ***DE*** of $\triangle DEF$. The same relationship is true for points ***V*** and ***T***, which are directly opposite vertices ***D*** and ***E***, respectively (Figure 6).

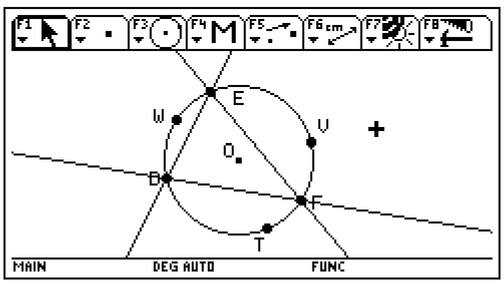

Figure 6

4. The two triangles will be similar (Figure 7).

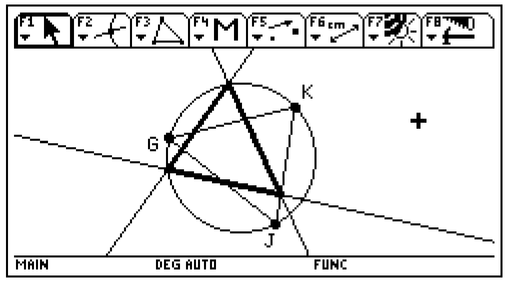

Figure 7

Explore:

1. The envelope of the Simson lines for a given triangle is a **deltoid**. Figure 8 shows this deltoid drawn as the locus of Simson lines. This view is with the **Link Locus Points** and **Envelope of Lines** options turned **On** on the **Geometry Format** menu (◆ **F**). Figure 9 shows an alternate way to view this relationship with the **Envelope of Lines** options turned **Off**.

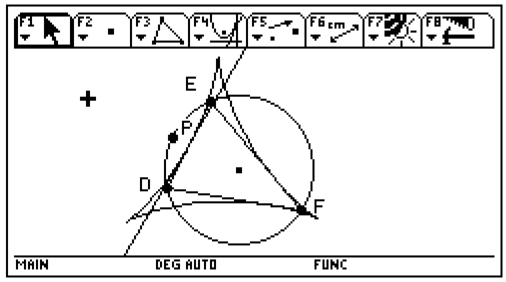

Figure 8

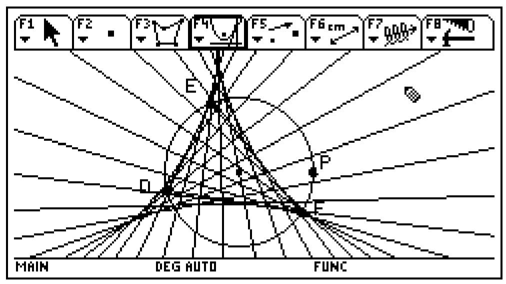

Figure 9

2. Let point ***Q*** be diametrically opposite point ***P*** on the circumcircle of $\triangle DEF$. The two Simson lines defined for points ***P*** and ***Q*** are perpendicular and meet on the nine-point circle of $\triangle DEF$ (Figures 10 and 11).

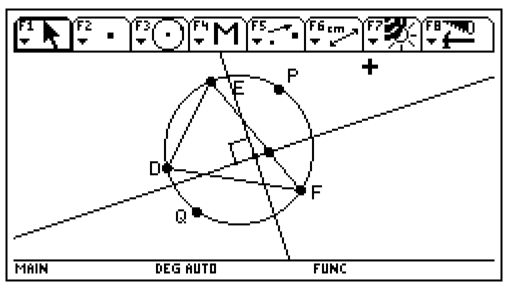

Figure 10

The locus of points of the intersection of two diametrically opposite points on the circumcircle is the nine-point circle of $\triangle DEF$. As shown in Figure 11, this circle passes through the midpoints of the sides of the triangle (three points), the feet of the altitudes (three points), and the midpoints of the segments connecting the triangle's vertices with its orthocenter (three points).

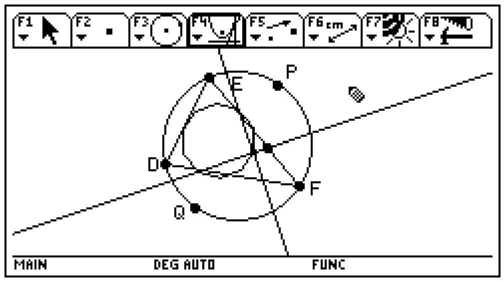

Figure 11

The center of the nine-point circle is the midpoint ***N*** of the **Euler segment**, connecting the orthocenter ***H*** and circumcenter ***O*** of a triangle (Figure 12).

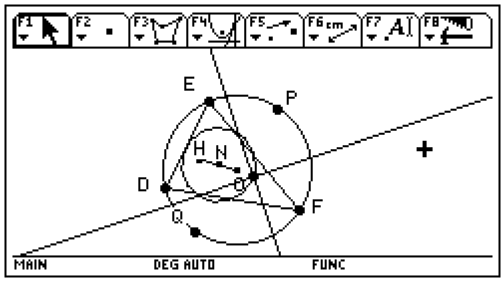

Figure 12

In some references, the nine-point circle is referred to as the **Feuerbach circle** for Karl Feuerbach (1800–1834). Although Feuerbach was not the first to discover this circle, his work defined many of the properties of this interesting construction. For example, **Feuerbach's Theorem** states that the nine-point circle of a triangle is tangent to the incircle and each of the three excircles of a triangle (Smart, 1978, p. 335).

As shown in Figure 13, the Feuerbach circle of the original triangle is inscribed in the deltoid that is the locus of the envelope of Simson lines for a given triangle. The deltoid is inscribed in a circle that has a radius three times and an area nine times that of the Feuerbach circle. Drag the vertices of your triangle around the screen to see whether this relationship is true for any triangle. As with all Feuerbach circles, its radius is half the radius of the circumcircle, and therefore, its area is one-fourth the area of the circumcircle. The area of the Feuerbach circle is half the area of the deltoid defined by the envelope of Simson lines for any given triangle.

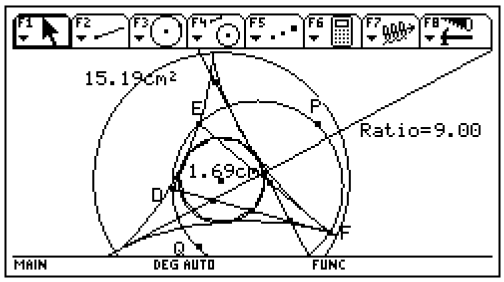

Figure 13

