Exponential Growth

by

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Textbook Correlation: Key Topic

- Growth and Decay

NCTM Principles and Standards:

- Process Standard
  - Representation
  - Connections
  - Problem Solving

Exercise 1.

Suppose a population of ants numbered 3,000,000 in 1998 and 3,300,000 in 1999. The ratio of the current generation to the previous generation $\frac{3{,}300{,}000}{3{,}000{,}000} = 1.1$ indicates that the net population growth for the first year is 10%.

a) Assuming the growth rate will be the same every year, what will the population figures be for several years?

b) Assuming the growth rate will be the same every year, what will the population be after 100 years?

Solution:

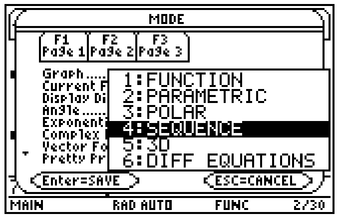
Although this can be iterated easily by hand or simple repeated multiplication on most calculators, the exploration can be done on the TI-89 (TI-92 Plus).

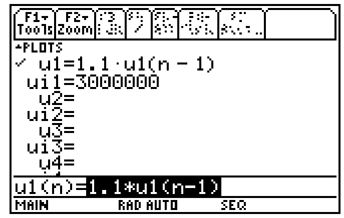
Given the initial population is 3,000,000, the model is described by the recursion formula:
$p_0 = 3{,}000{,}000$
$p_n = 1.1\ p_{n-1}$, where n = 0,1,2,3…years and p_n is the number of ants after n years.

The folllowing analysis includes three methods: numerical, graphical and analytical.

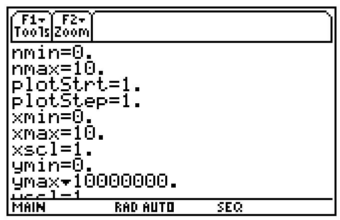
Numerical Analysis

First, press **MODE** Cursor to the right of "Graph" and select **4: SEQUENCE**. Press **ENTER** twice. Next, go to the Y= Editor (♦, **F1**). For u1(n), type **1.1u1(n-1)** and press **ENTER**. For ui1, type **3000000** and press **ENTER**.

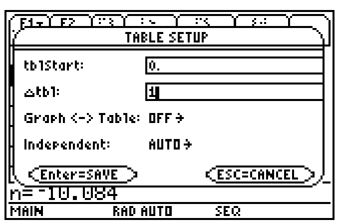


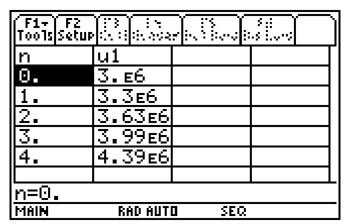


Set the parameters in the **WINDOW** Editor (♦, **F4**)as shown below.

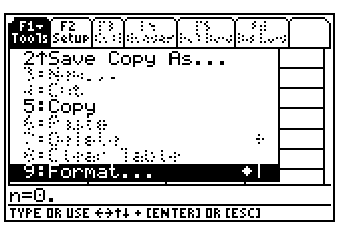


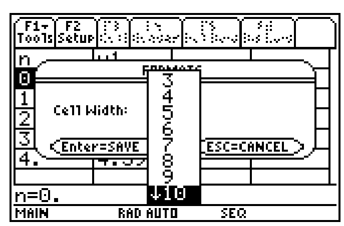
To analyze the model numerically, press ♦, **F4** for **TblSet**. Enter **0** for "tblStart" and **1** for "Δ table." Remember to press **ENTER** twice.

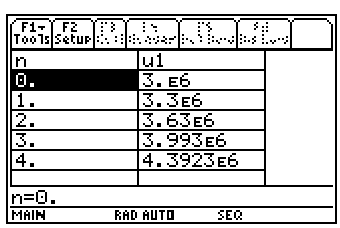




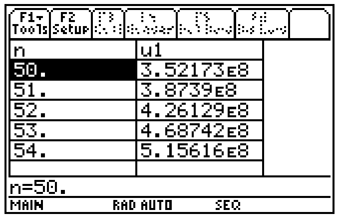
To widen the width of the cells, press **F1**(Tools), **9:Format**, cursor right, choose the cell width and press **ENTER**.

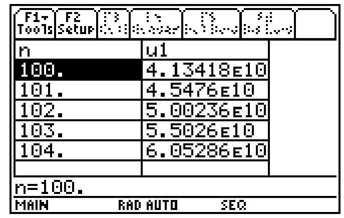






Cursor down through the table. Pressing **2^{nd}** and the down cursor key will move down one page at a time. To observe long time behavior reset the “tblStart” for large values to save time.

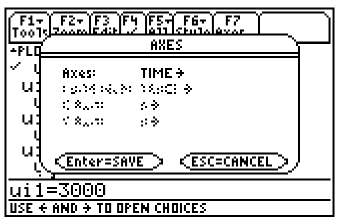


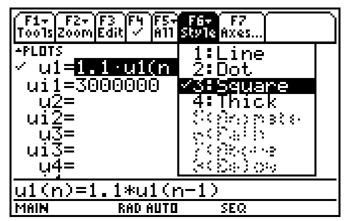


From the limited information on the table, it appears that the population is increasing without bound.

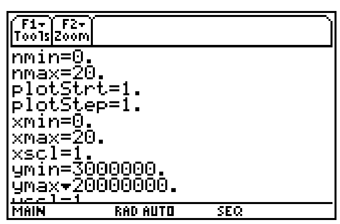
Graphical Analysis

To look at a graphical representation of the model, go to the **Y=** Editor (♦, **F4**). Press **2^{nd}**, **F2** for **F7** (Axes). Select **TIME**. Highlight the equation to the right of u1. Press **2^{nd}**, **F1** for **F6** (Style). Select 2:Dot or 3:Square.

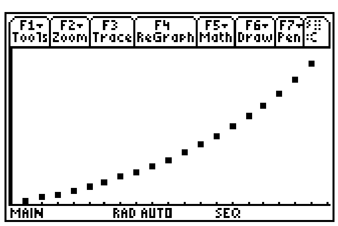


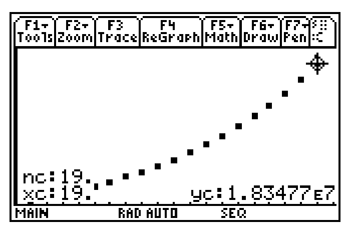


Select 1:Line. Next, go to the Window (♦, **F2**). Set the parameters as illustrated below.



Now **GRAPH** the model (♦, **F2**). Trace (**F3**).





From the graph it appears that the population is increasing without bound. Are you convinced?

Analytical Solution

Rewriting the model p_0 = 3,000,000, p_n = 1.1 p_{n-1} in function notation yields
p(0) = 3,000,000, p(n) = 1.1p(n-1).
Observe that

$$p(0) = 3{,}000{,}000$$
$$p(1) = 1.1p(0) = 1.1(3{,}000{,}000)$$
$$p(2) = 1.1p(1) = 1.1[1.1\,p(0)] = 1.1^2\,p(0) = 1.1^2\,(3{,}000{,}000)$$
$$p(3) = 1.1p(2) = 1.1[1.1^2 p(0)] = 1.1^3\,p(0) = 1.1^3\,(3{,}000{,}000)$$
.
.
.
$$p(k) = 1.1p(k-1) = 1.1[1.1^{k-1}p(0)] = 1.1^k\,p(0) = 1.1^k\,(3{,}000{,}000)$$

In general, we make the conjecture:

The analytical solution of $p(n) = r\,p(n-1)$, n = 0,1,2…., is $p(n) = r^n\,p(0)$.

Verify the conjecture as follows:

Given	$p(n) = r\,p(n-1)$,
substitute	$r^n\,p(0) = r\,[r^{n-1}\,p(0)]$.
Therefore,	$r^n\,p(0) = r^n\,p(0)$.

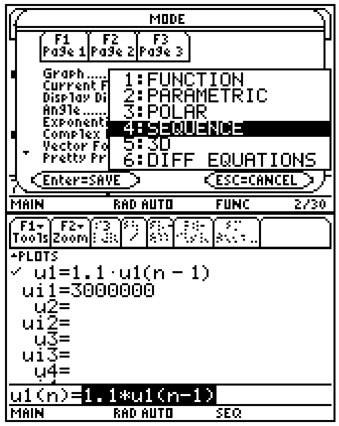
A model like this one with the population growing by the same rate each year is called an **exponential model**. It is described by a recursive equation and an explicit equation.
The recursive equation is of the form $p_n = rp_{n-1}$, where r is a constant called the population multiplier. The analytical solution, $p(n) = r^n\,p(0)$, where p(0) is the initial population, is an explicit equation describing the same sequence. It is a geometric sequence with a common ratio, r. Exponential models represent positive growth when r is poitive or negative growth (decay) when r is negative. Populations can be comprised of animate or inanimate objects.

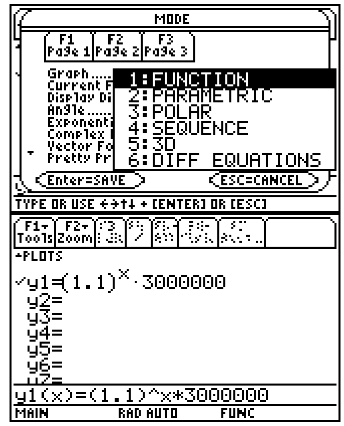
Comparison of the Recursive Description and the Explicit Description of an Exponential Growth Model

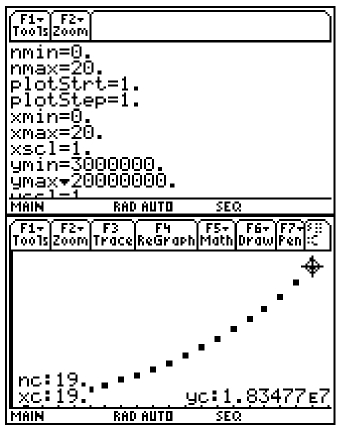
In the previous example, the analytical solution to
p(0) = 3,000,000 p(n) = 1.1p(n-1), n = 0,1,2… is
$p(n) = (1.1)^n\,(3{,}000{,}000)$.

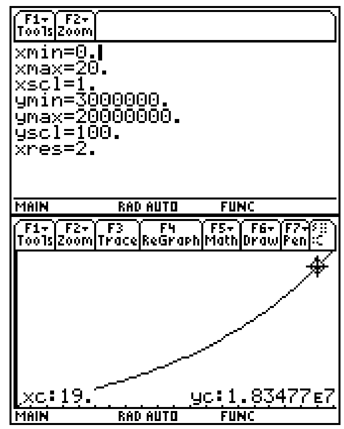
Graph the analytical solution in function mode on the TI-89. Press MODE, 1:Function, ENTER, ENTER. In the **Y=** editor (♦, **F1**), enter **1.1^x*3000000**. Set the window (♦, **F2**) as depicted below. Graph (♦, **F3**), trace (**F3**, **9**) and look at the table (♦, **F5**).

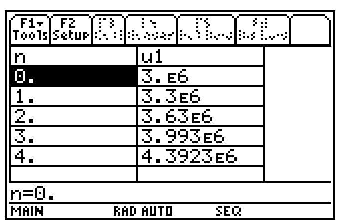
The following screens show the comparison of the recursive description on the left and its analytical solution, an explicit description, on the right.

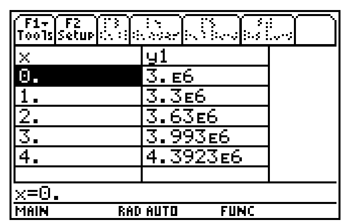

MODE
F1 Page 1 F2 Page 2 F3 Page 3
1:FUNCTION
2:PARAMETRIC
3:POLAR
4:SEQUENCE
5:3D
6:DIFF EQUATIONS
Enter=SAVE
ESC=CANCEL
MAIN RAD AUTO FUNC 2/30
PLOTS
u1=1.1·u1(n − 1)
ui1=3000000
u2=
ui2=
u3=
ui3=
u4=
u1(n)=1.1*u1(n-1)
MAIN RAD AUTO SEQ

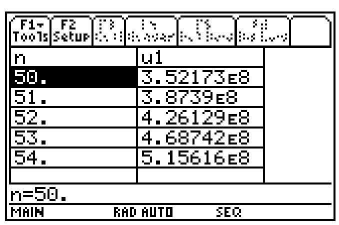

MODE
F1 Page 1 F2 Page 2 F3 Page 3
1:FUNCTION
2:PARAMETRIC
3:POLAR
4:SEQUENCE
5:3D
6:DIFF EQUATIONS
Enter=SAVE
ESC=CANCEL
TYPE OR USE ←→↑↓ + [ENTER] OR [ESC]
PLOTS
y1=(1.1)^x·3000000
y2=
y3=
y4=
y5=
y6=
y1(x)=(1.1)^x*3000000
MAIN RAD AUTO FUNC

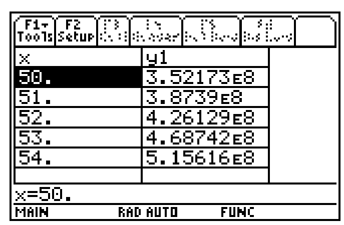

nmin=0.
nmax=20.
plotStrt=1.
plotStep=1.
xmin=0.
xmax=20.
xscl=1.
ymin=3000000.
ymax=20000000.
MAIN RAD AUTO SEQ
F1 Tools F2 Zoom F3 Trace F4 ReGraph F5 Math F6 Draw F7 Pen
nc:19.
xc:19.
yc:1.83477E7
MAIN RAD AUTO SEQ

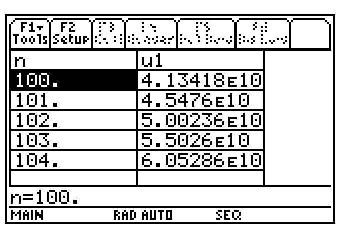

xmin=0.
xmax=20.
xscl=1.
ymin=3000000.
ymax=20000000.
yscl=100.
xres=2.
MAIN RAD AUTO FUNC
F1 Tools F2 Zoom F3 Trace F4 ReGraph F5 Math F6 Draw F7 Pen
xc:19.
yc:1.83477E7
MAIN RAD AUTO FUNC

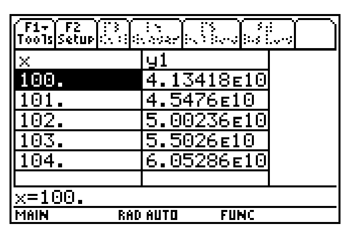

n u1
0. 3.E6
1. 3.3E6
2. 3.63E6
3. 3.993E6
4. 4.3923E6
n=0.
MAIN RAD AUTO SEQ


x y1
0. 3.E6
1. 3.3E6
2. 3.63E6
3. 3.993E6
4. 4.3923E6
x=0.
MAIN RAD AUTO FUNC


n u1
50. 3.52173E8
51. 3.8739E8
52. 4.26129E8
53. 4.68742E8
54. 5.15616E8
n=50.
MAIN RAD AUTO SEQ


x y1
50. 3.52173E8
51. 3.8739E8
52. 4.26129E8
53. 4.68742E8
54. 5.15616E8
x=50.
MAIN RAD AUTO FUNC


n u1
100. 4.13418E10
101. 4.5476E10
102. 5.00236E10
103. 5.5026E10
104. 6.05286E10
n=100.
MAIN RAD AUTO SEQ


x y1
100. 4.13418E10
101. 4.5476E10
102. 5.00236E10
103. 5.5026E10
104. 6.05286E10
x=100.
MAIN RAD AUTO FUNC

Answer:

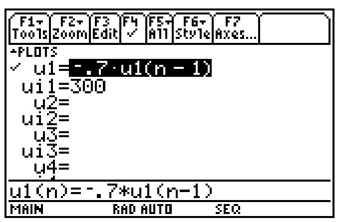
a) The population in 1998 = p(0) = $(1.1)^0$ (3,000,000) = 3 x 10^6.
The population in 1999 = p(1) = $(1.1)^1$ (3,000,000) = 3.3 x 10^6.
The population in 2000 = p(2) = $(1.1)^2$ (3,000,000) = 3.63 x 10^6.
The population in 2001 = p(3) = $(1.1)^3$ (3,000,000) = 3.99 x 10^6.
The population in 2002 = p(3) = $(1.1)^4$ (3,000,000) = 4.3923 x 10^6.

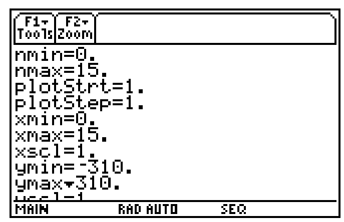
b) After 100 years the population will be p(100) = $(1.1)^{100}$ (3,000,000) = 4.13418 x 10^{10}.

Exercise 2:

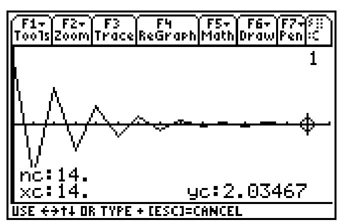
Consider the model p(0) = 300, p(n) = r×p(n-1) where n= 0,1,2,…
The explicit description of the model is p (n) = 300 r^n.
Experiment to observe results when |r| > 1, r = 1, and |r|< 1. Let r = 0.8, 1.8, -0.8, -1.8.
What is the long term behavior of the models ?

For example, let r = -0.7:

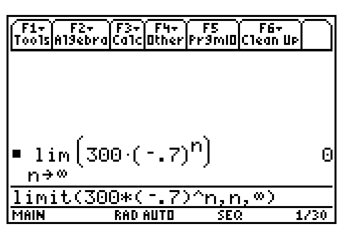




Note: Select **1:Line** under **Style** (**F6**) in the **Y=Editor** to follow the sequence.



The explicit equation is p(n) = $300(-0.7)^n$. The limit of p(n) as n approaches infinity is zero ($\lim_{n\to\infty} 300(-0.7)^n = 0$).



Answer for this example: When r = -0.7, the sequence oscillates with the amplitude of the oscillations decreasing as the sequence approaches 0.

Exercise 3:

Change the initial condition. Does the change in initial condition affect the long term behavior of the model?