An Investigation in Calculus using the TI-92

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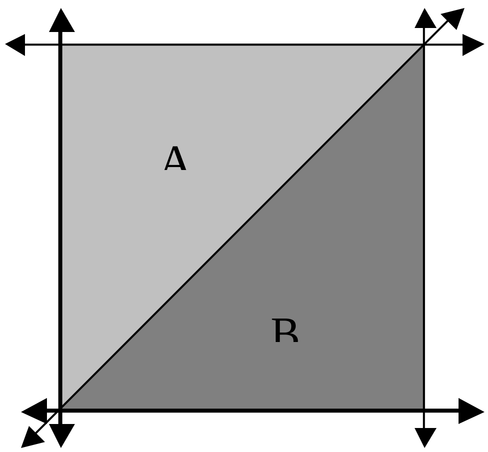
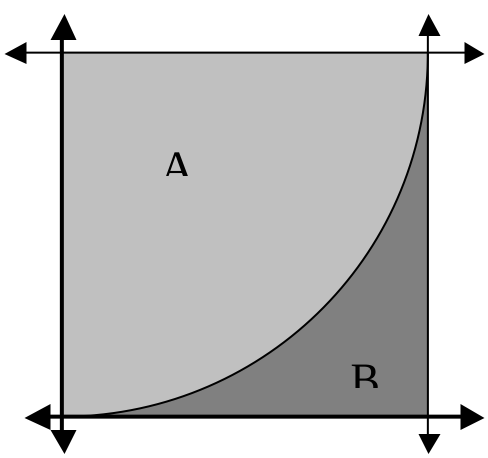
We start with the problem as follows. Given the graphs of y = x, y =1 and x=1:



We then ask the simple question: What is the ratio of the Area A to the Area B? By straightforward calculation of the areas of the two triangles we get the answer 1 : 1.

The problem starts to become more interesting if we consider the graphs y=x^2, y=1 and x=1.



The area B is calculated, by hand, as $\int_0^1 x^2 dx$ which is 1/3 and, since the area of A is calculated by subtracting the area of B from a unit square, we get that the area of A is 2/3. The ratio of the areas is therefore 2 : 1.

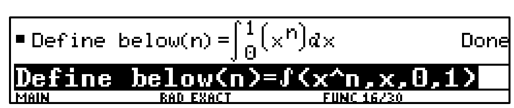
Following the same steps for y = x^3 we find that the area is 3 : 1. We conjecture at this point that there is a relationship between the index of x in the curve and the ratio of Area A : Area B. This ratio can be calculated as follows:

$$\frac{1-\int_0^1 x^n dx}{\int_0^1 x^n dx} = \frac{1-\left[\frac{x^{n+1}}{n+1}\right]_0^1}{\left[\frac{x^{n+1}}{n+1}\right]_0^1}$$

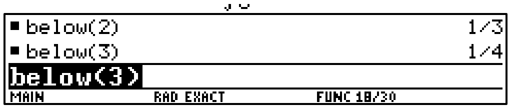
$$= \frac{1-\frac{1}{n+1}}{\frac{1}{n+1}} = \frac{\frac{n}{n+1}}{\frac{1}{n+1}} = \frac{n}{1}$$

We can therefore see that in general the ratio of Area A : Area B is n : 1.

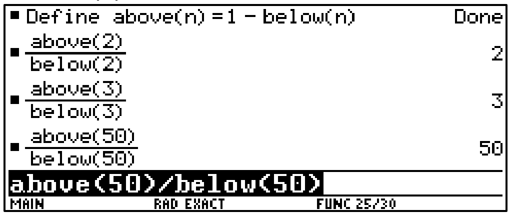
We would like to develop the problem in a number of ways, in particular by changing the limits of integration. We would like to use the TI-92 for this problem and so, as a first step, we will check our work thus far on the TI-92 by defining some functions to calculate the areas A and B.



Whenever a function is defined on the TI-92, it is always a good idea to test the function on a few known results:



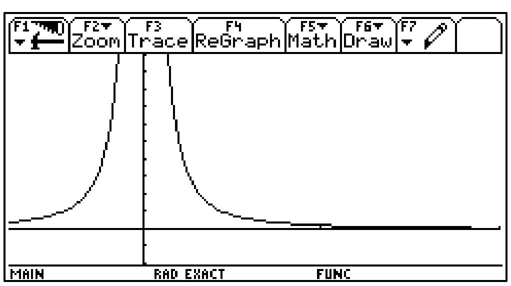
All seems well. We define above(n) and test some results:



And now for the interesting result:

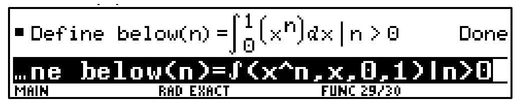
What????

Look at all the examples we have seen so far: n= 2, 3, 4, 50. All of these are positive integers. Of course the original question was posed with n implicitly meant to be a positive integer. A moment's thought about the graph of y = x^n with x a negative integer reveals that there is no area enclosed under the graph between 0 and 1 as the following graph of x^(-2) suggests:



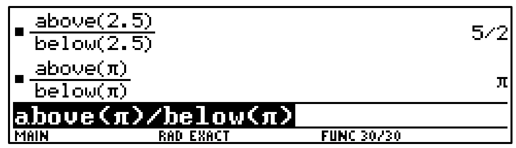
This is a good example of how the TI-92 promotes exact mathematical thinking. Our casual definition of the function below(n) was not quite what we meant but the TI-92 forces us to think more clearly.

We need to modify our definition of below(n) using the "with" (|) operator:



Now we go back to our question:

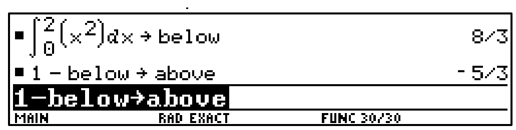
In fact, the problem works for any real number greater than 0 not just integers:



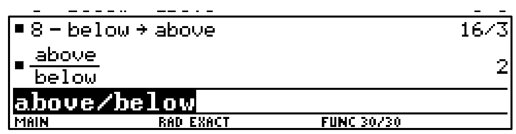
Extension 1

A reasonable first extension to this problem is to ask what happens if we integrate from 0 to 2. Before we start again we want to delete the functions below and above:

We start again from the basics and can generate the first results by hand or with the TI-92 as follows:



Not very promising, but of course the total of area A and area B is no longer a square. The value of the function f(x) = x^2 at x = 2 is 4 and so we are looking at a two by four rectangle with area 8.



If the function is f(x) = x^3 then the shape is a two by eight rectangle with area 16:

```
■ ∫[0,2](x^3)dx → below                         4
■ 16 - below → above                            12
■ above/below                                   3
above/below
MAIN        RAD EXACT        FUNC 30/30
```

In general the value of f(x) = x^n at x = 2 is 2^n, so the rectangle we need to consider for the total area is 2 * 2^n or 2^(n+1). So proceeding as before:

```
■ Define below(n) = ∫[0,2](x^n)dx | n > 0       Done
■ Define above(n) = 2^(n+1) - below(n)          Done
…ne above(n)=2^(n+1)-below(n)
MAIN        RAD EXACT        FUNC 30/30
```

We check a few known values as before:

```
■ below(2)                                      8/3
■ above(3)/below(3)                             3
above(3)/below(3)
MAIN        RAD EXACT        FUNC 30/30
```

And finally:

```
■ above(n)/below(n)                             n
above(n)/below(n)
MAIN        RAD EXACT        FUNC 30/30
```

Extension 2

So if this property holds while integrating from 0 to 1 and integrating from 0 to 2, will it work from 0 to b, an arbitrary parameter?

```
■ Define below(n,b) = ∫[0,b](x^n)dx | n > 0     Done
… below(n,b)=∫(x^n,x,0,b)|n>0
MAIN        RAD EXACT        FUNC 30/30
```

We check this definition for known values:

```
■ below(2,1)                                    1/3
■ below(3,2)                                    4
below(3,2)
MAIN        RAD EXACT        FUNC 30/30
```

The value of the function f(x) = x^n at x = b is b^n. The rectangle under consideration is then b*b^n or b^(n+1) :

```
■ Define above(n,b) = b^(n+1) - below(n,b)
                                                Done
…bove(n,b)=b^(n+1)-below(n,b)
MAIN        RAD EXACT        FUNC 30/30
```

And we look for the general result:

```
■ above(n,b)/below(n,b)                         n
above(n,b)/below(n,b)
MAIN        RAD EXACT        FUNC 30/30
```

And so the property holds on the integral from 0 to an arbitrary parameter b.

Extension 3

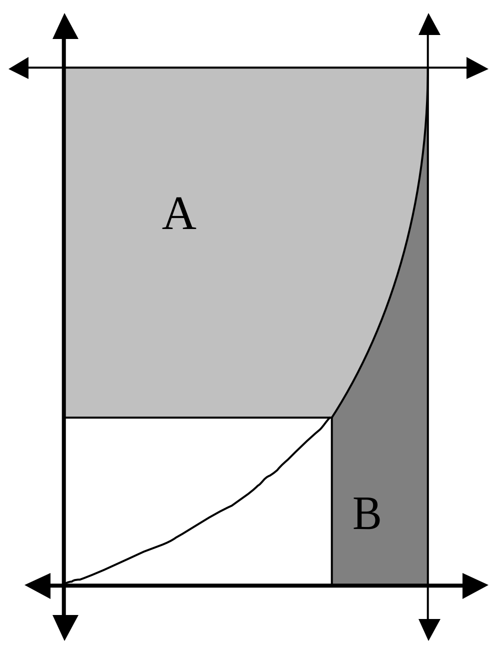
The next question that arises is: if it works for 0 to 1 and it works for 0 to 2, does it work from 1 to 2?

As before we delete `below` and `above`:

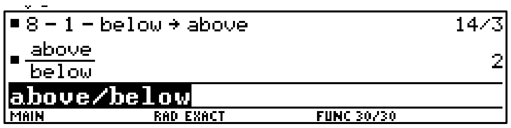
```
■ DelVar above,below                            Done
delvar above,below
MAIN        RAD EXACT        FUNC 30/30
```

To find the new area B we take:

```
■ ∫[1,2](x^2)dx → below                         7/3
∫(x^2,x,1,2)→below
MAIN        RAD EXACT        FUNC 30/30
```

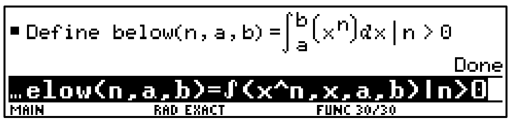


As before the total area is a 2 by 4 rectangle with area 8 and there is a unit square left unshaded. So the area A is 8 - 1 - 7 / 3

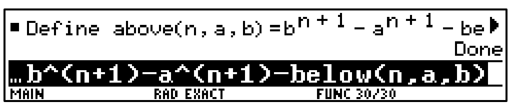


And again the property seems to hold in this case.

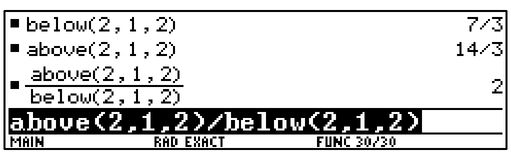
We now turn our mind to the most general result which is for the curve x^n, considered between arbitrary parameters a and b with a < b:



The values of the function f(x) = x^n at a and at b are a^n and b^n respectively. The total area of the shape is b*b^n or b^(n+1) and the unshaded part is a*a^n or a^(n+1). Therefore:



Once more we check the definitions for some known values:



Finally, the general result:

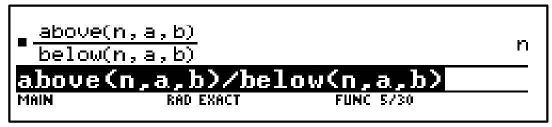
Extension 4

This entire investigation can be expanded to the question of volumes of revolution. It works but doesn't have the same answer as with the area and is dependent on the axis of revolution.

Extension 5

To return finally to the question of a negative value of n. The property does, in fact hold for values of n so long as the limits of integration are either both positive or both negative. In other words the problem is if one of the endpoints of the integral is 0 or the integration is across 0.

To my surprise however, when we define the functions `below(n,a,b)` and `above(n,a,b)` with no restriction on n we get



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