



Math Objectives

- Students will model the spread of a rumor using a recursive sequence.
- Students will understand that variations in the recursive sequence modeling the spread of a rumor or a disease can lead to predictable or chaotic behavior.
- Students will model with mathematics (CCSS Mathematical Practice).
- Students will use appropriate tools strategically (CCSS Mathematical Practice).

Vocabulary

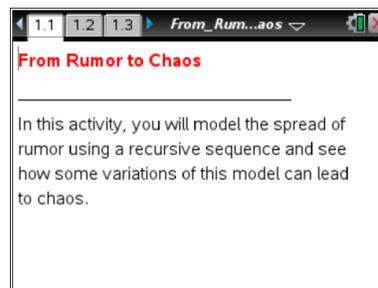
- recursive sequence

About the Lesson

- This lesson involves modeling the spread of a rumor and similar problems.
- As a result, students will:
 - Find and analyze a recursive sequence modeling the spread of a rumor.
 - Explore variations in this sequence that lead to predictable and chaotic behavior

TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to monitor student progress.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- Once a function has been graphed, the entry line can be shown by pressing **ctrl** **G**. The entry line can also be expanded or collapsed by clicking the chevron.

Lesson Files:

Student Activity

From_Rumor_to_Chaos_Student.pdf

From_Rumor_to_Chaos_Student.doc

TI-Nspire document

From_Rumor_to_Chaos.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.



Discussion Points and Possible Answers

Terry loves to gossip. After hearing a rumor, Terry wants all of his/her 24 friends to hear the same rumor. Terry's mother insisted that Terry can make only one phone call per day and the same phone-call restriction is true of all of Terry's friends. Assume:

- A. Terry hears a rumor on day 1. On day 2, Terry randomly calls and reveals the rumor to a friend.
- B. On day 3, 4, ... , each person who already knows the rumor randomly calls one of the other 24 students and reveals the rumor (if that person had not already heard it).
- C. The process is repeated daily until all 25 students (Terry and friends) in the class know the rumor.

Terry records the total number of people who know the rumor each day. On any given day, some people who already know the rumor might be called and told again. These people will not be counted twice. The data are given in the following table:

Day (n)	Number who knew (u_n)
1	1
2	2
3	4
4	8
5	13
6	20
7	24
8	25

Next, you will find a recursive sequence to model this data. To develop such a model, we make the assumption that:

The number of people who learn the rumor on a given day is proportional to the product of the number of people who know the rumor and the number who don't.

1. Explain why this assumption is reasonable.

Sample Answers: The product represents the number of "interactions" (in this case phone calls) between those who know the rumor and those who don't. A certain fraction of these interactions will result in more students knowing the rumor.



We need to represent the following quantities in the model:

- (1) the number of people who know the rumor at any given time.
- (2) the number of people who do not know the rumor at any given time.
- (3) the total population – or a reasonable estimate for this value.
- (4) the rate at which the rumor spreads.

Let u_n denote to the number of people who know the rumor at the end of n days, P denote the total population, and k the constant of proportionality. Because of the assumption above, a reasonable recursive sequence model is

$$u_n = u_{n-1} + k \cdot u_{n-1} \cdot (25 - u_{n-1})$$

This model is an example of the general recursion model: *new value = old value + change*.

In this case, we know $P = 25$, so we must determine a reasonable value of k .

Teacher Tip: You could mention that this problem is simplistic and not very realistic. More sophisticated versions of the problem, however, are used to model the spread of a disease or illness and other things similar to rumors.

Move to page 1.2.

To find a reasonable value for k in this model, we first rewrite the recursive sequence as $u_n - u_{n-1} = ku_{n-1} \cdot (25 - u_{n-1})$ —an equation of the form $y = kx$ where x is the product $ku_{n-1} \cdot (25 - u_{n-1})$ and y is the difference $u_n - u_{n-1}$.

day	num	cha	prod
1	1	1	24
2	2	2	46
3	3	4	84
4	4	8	136
5	5	13	156

The following procedure was used to create the spreadsheet on Page 1.2. You will need to use this procedure for Problem 2 or, perhaps, on this problem if a different data set is used.

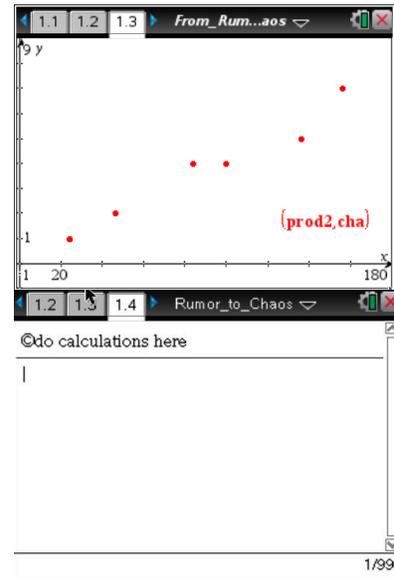
- Enter the data into Columns A and B of the table: *day* and *num*.
- Automatically calculate values for Column C (named *cha*) by moving your cursor to the formula cell directly under the name of the column and selecting **MENU > Data > List Operations > Difference list [Δ List] (num)**.
- Calculate values for Column D (named *prod*) by typing $= num \cdot (25 - num)$ in the formula cell under the name of the list.
- In Column E (named *prod2*), type $= left(prod, dim(cha))$. This formula ensures that the lists *cha* and *prod2* have the same number of elements.



Teacher Tip: Instead of using the data in the table, you could simulate a data set using $randInt(1, N)$ where N is the class size. Assign each student a number from 1 to N . Choose one random integer – that student is the one who knows the rumor. Then choose another random integer – the first student who is called. Assuming this number is different from the first one, then choose two random integers representing the next two students who are called. Continue until all the integers have been generated and construct a table similar to the one above.

Move to page 1.3.

A scatter plot using $x = prod2$, $y = cha$ has been drawn on this page.



Move to page 1.4.

- Find a suitable value of k . Hint: Because the model has the form $y = kx$, one method is associated with linear regression.

$$k = \underline{\hspace{2cm}}$$

Sample Answers: Using linear regression [MENU > Statistics > Stat Calculations > Linear Regression (mx+b)] gives $y = 0.041436x + 0.054475$, so $k = 0.0414$ is reasonable since the y-intercept is small [.055]. If students know the formula for finding k for proportional models of the

form $y = kx$: $k = \frac{\sum x_i y_i}{\sum x_i^2}$ where the data set is $\{(x_i, y_i)\}$, then $k = 0.0419$.

Teacher Tip: You could mention that this problem illustrates of the use of the method of “Transform-Solve-Invert”. The data is first “transformed” [in this case by rewriting the equation and identifying the variables x and y in $y = kx$], the problem is solved [use linear regression], and then inverted back the original form [rewriting in original form $u_n = u_{n-1} + ku_{n-1} \cdot (25 - u_{n-1})$]. More familiar examples include performing exponential regression by using linear regression on the transformed data (x, y) to $(x, \log y)$ or power regression by using linear regression on the transformed data (x, y) to $(\log x, \log y)$.

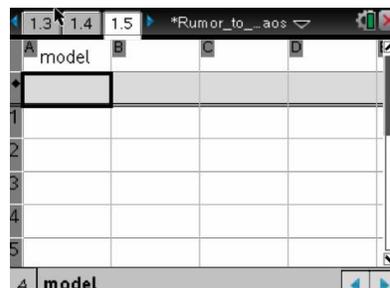
- Using this value of k , write your model $u_n = \underline{\hspace{2cm}}$.



Sample Answers: $u_n = u_{n-1} + 0.0414u_{n-1} \cdot (25 - u_{n-1})$ which can be written as $u(n) = u(n-1) + 0.0414 * u(n-1) * (25 - u(n-1))$.

Move to page 1.5.

- On this spreadsheet page, enter your answer to Question 3 as a sequence in the formula cell – the second cell of the Column A [model] by selecting **MENU > Data > Generate Sequence**, typing $u(n-1)$ for u_{n-1} , etc. and setting *initial terms* = 1 and *nMax* = 15.



- According to your model, how many students knew the rumor at the end of the sixth day?

Sample Answers: Using the model of Question 3: $u_6 = 19.108$ or 19 students. Note the value in the original data set is 20 students.

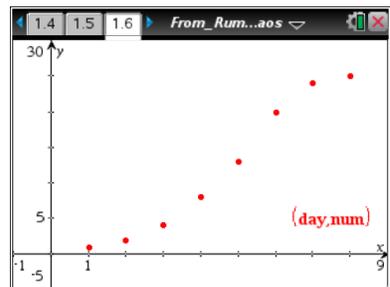
- According to your model, between which two days was the increase in the number of students knowing the rumor the largest?

Sample Answers: Between days 5 and 6 – an increase of 6 or 7 students. Use **MENU > Data > List Operations > Difference list [Δ List]** where [Δ List(model)] = 6.48 in this case.

Move to page 1.6.

A scatter plot of the original data is shown on Page 1.6 using $x = \text{day}$ and $y = \text{num}$.

- Draw a scatter plot of the data generated by the model using $x = \text{day}$ $y = \text{model}$. How do the two scatter plots compare?



Sample Answers: The two plots are essentially the same with the values of model data for days 5 and 6 being slightly smaller than those for the original data.



Move to page 2.1.

Problem 2—Another Rumor Problem

In this example, Terry has more friends, but we are not sure how many. The value of P is unknown and must be estimated. The other assumptions given in the previous problem are the same. Terry recorded the data in the table below.

day	num
1	1
2	2
3	4
4	8
5	15

Day (n)	Number who knew (u_n)
1	1
2	2
3	4
4	8
5	15
6	28
7	47
8	72
9	92
10	98

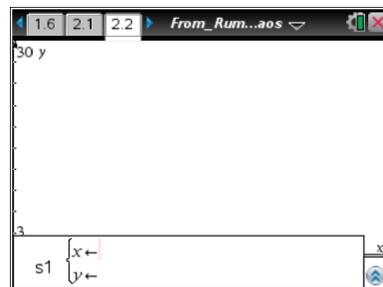
7. Using this data, find a reasonable estimate for the value of P in the model.

Sample Answers: $P = 100$ is reasonable.

- On Page 2.1, the data has been entered in the first two columns of the spreadsheet – *day* and *num*. Using the procedure from Question 1, create the three additional columns *cha*, *prod*, *prod2* using the value of P from Question 7.

Move to page 2.2.

Draw a scatter plot on Page 2.2 using $x = \textit{prod2}$ and $y = \textit{cha}$.

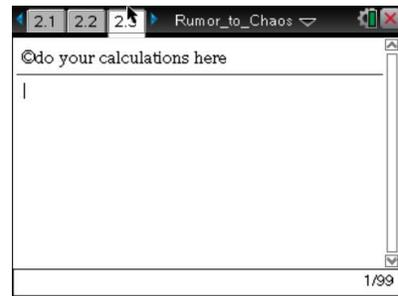




Move to page 2.3.

8. Find a suitable value of k . Hint: As in Question 2, one method involves linear regression.

Sample Answers: Using linear regression [Statistics > Stat Calculations > Linear Regression (mx+b)] gives $y = 0.0099x - 0.17772$ so $k = 0.0099$ is reasonable since the y-intercept is small [-0.1777]. If students know the formula for finding k for proportional models of the form $y = kx$: $k = \frac{\sum x_i y_i}{\sum x_i^2}$ where the data set is $\{(x_i, y_i)\}$, then $k = 0.0098$.



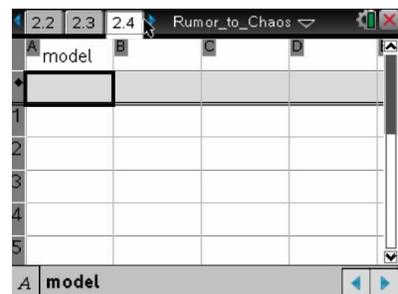
9. Using this value of k , write your model $u_n =$ _____ .

Sample Answers: $u_n = u_{n-1} + .0099u_{n-1}(100 - u_{n-1})$

TI-Nspire Navigator Opportunity: Screen Capture
See Note 1 at the end of this lesson.

Move to page 2.4.

- Enter your answer to Question 9 as a sequence in the first column of the spreadsheet using the process from Question 4.
10. According to your model, how many students knew the rumor at the end of the sixth day?



Sample Answers: Using the model of Question 9: $u_6 = 26.9$ or 27 students. Note the value in the original data set is 28 students.

11. According to your model, between which two days was the increase in the number of students knowing the rumor the largest?

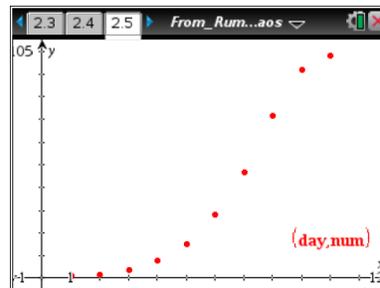
Sample Answers: Between days 7 and 8 – an increase of 24 or 25 students ($\Delta \text{List}(\text{model}) = 24.62$).



Move to page 2.5.

A scatter plot of the original data is shown on Page 2.5 using $x = \text{day}$ $y = \text{num}$

12. Draw a scatter plot of the data generated by the model using $x = \text{day}$ $y = \text{model}$. How do the two scatter plots compare?



Sample Answers: The two plots are essentially the same.

Move to page 3.1.

Problem 3

We will now consider a modified (logistic) model of the one used in Problems 1 and 2.

A spreadsheet window titled 'From_Rum...aos' showing a table with two columns: 'num' and 'val'. The data is as follows:

	num	val
1	1	0.6
2	2	0.492
3	3	0.512369
4	4	0.512186
5	5	0.512106

At the bottom of the spreadsheet, there is a slider control labeled 'k = 2.05'.

Imagine that an illness is spreading through your school. Let u_n denote the fraction of students who have the illness after n days so that $(1 - u_n)$ is the fraction of the students who do not have the illness. Both u_n and $(1 - u_n)$ are between 0 and 1 and the total population is now $P = 1$. We have the model $u_n = ku_{n-1}(1 - u_{n-1})$ that reflects the assumption that the “new” fraction of those having the illness, u_n , is proportional to the product of the “old” fraction who have the illness, u_{n-1} , and the fraction who do not have the illness, $(1 - u_{n-1})$. The interaction between the two groups will change the fraction representing those who have the illness at time increases.

This model is most interesting. For some values of k , the behavior of the values of u_n varies from predictable to chaotic. In this problem, we consider values of k for which $2 \leq k \leq 4$.

On Page 3.1, a spreadsheet has been created with two columns: *num* and *val* [value]. The entries in *num* are integers from 1 to 100. The entries in *val* are the terms of the sequence $u_n = ku_{n-1}(1 - u_{n-1})$ for values of k between 2 and 4 that you select using the clicker below the spreadsheet.

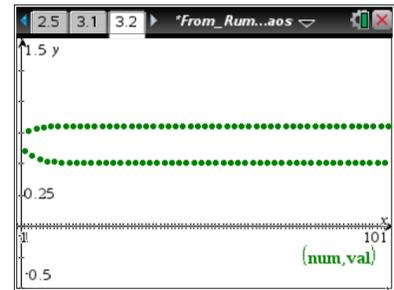


Move to page 3.2.

On Page 3.2, a scatter plot with $x = \text{num}$ and $y = \text{val}$ has been created. The example with $k = 3.2$ and $u_0 = .6$ is shown. Consider various values for k , and toggle between Pages 3.1 and 3.2.

Explore the behavior of the sequence $u_n = ku_{n-1} \cdot (1 - u_{n-1})$ and the corresponding scatter plot for various values of k and various initial values u_0 where $0 < u_0 < 1$. To change the value of u_0 ,

change the value in cell B1 on Page 3.1. The rest of the entries in this column and corresponding scatter plot on Page 3.2 are automatically updated.



13. To examine the values in the scatter plot on Page 3.2, select **MENU > Trace > Graph Trace**. There are four basic types of behavior that occur. Describe each of them, and give the approximate range of values of k for which each occurs.

Sample Answers: The set of sequence values of k that can be accessed by the k -clicker are divided into several groups. The four basic types of behavior are: a single convergent sequence, two groups of sequence values, three groups of sequence values, and sequence values that are chaotic and have no apparent pattern.

$2 \leq k \leq 3$: For a fixed value of k in this range and all values of u_0 , $0 < u_0 < 1$, the sequence converges to a limiting value.

$3.05 \leq k \leq 3.40$: For a fixed value of k in this range and all values of u_0 , $0 < u_0 < 1$, the values in the sequence are divided into two groups. Within each group, these values approach a limiting value while the sequence oscillates between the two groups.

$3.45 \leq k \leq 3.55$: For a fixed value of k in this range and all values of u_0 , $0 < u_0 < 1$, the values in the sequence are divided into two groups. Within each group, these values approach two or more limiting values while the sequence oscillates between the two groups. This behavior is not always apparent since only 100 terms of the sequence are graphed.

$3.6 \leq k < 4.0$ except 3.85: For a fixed value of k in this range and all values of u_0 , $0 < u_0 < 1$, the values in the sequence are chaotic or unpredictable

3.85: For a fixed value of k in this range and all values of u_0 , $0 < u_0 < 1$, the values in the sequence are divided into three groups. Within each group, these values approach two or more limiting values while the sequence oscillates between the three groups. This behavior is not always apparent since only 100 terms of the sequence are graphed.



TI-Nspire Navigator Opportunity: *Screen Capture*

See Note 2 at the end of this lesson.

Teacher Tip: Encourage students to conduct research about this “logarithmic map” and chaos. There are other interesting results for values of k not accessible by the k -clicker settings in this problem. Even those values of k that seem to indicate chaotic behavior might have some underlying patterns. See, for example, the Wikipedia entry for logarithmic maps.

Extension

Students could consider a continuous logistic model for the data in the two rumor problems. Two forms of logistic regression are available on the TI-Nspire handheld.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- How to find and analyze a recursive sequence modeling the spread of a rumor.
- How to analyze variations of a recursive sequence to determine predictable and chaotic behavior.

TI-Nspire Navigator

Note 1

Name of Feature: **Screen Capture**

Use Screen Capture to monitor student use of the spreadsheet and how they are finding an appropriate value for k for the recursive sequence in this second data set where most of the calculations must be performed by the student.

Note 2

Name of Feature: **Screen Capture**

Use Screen Capture to monitor student toggling back and forth between the spreadsheet on Page 3.1 and the scatter plot on Page 3.2. Mention that this single model exhibits many types of behavior including chaos – a rare occurrence in mathematics.