NUMB3RS Activity: Branch and Bound
Episode: "The Mole"

Topic: Linear Programming **Grade Level:** 9 - 12

Objective: Forcing integral solutions

Time: 20 minutes

Materials: TI-83 Plus/TI-84 Plus graphing calculator with Inequality Graphing App installed

Introduction

Charlie and Amita are trying to determine where an FBI mole will make the next information drop. Charlie proposes that "by running an algorithm that analyzes the features of the existing meeting places to find key variables, then applying a branching and bounding algorithm, I can hopefully give you a location for the mole's next probable destination."

Charlie is talking about a program to optimize the likelihood of an outcome. One method of optimization often studied in high school math courses is linear programming. The branch and bound algorithm Charlie mentions can be used when integer solutions are desired, but not attained.

This activity assumes that students already understand the ideas behind linear programming. Rather than introduce linear programming, this activity is meant to provide an area for further study and to discuss using the calculator to find the feasible region and points of intersection for a system of inequalities.

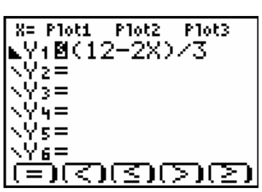
Discuss with Students

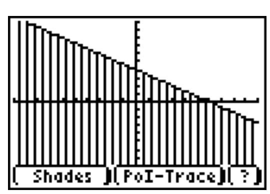
Consider the system below.

$$2x + 3y \le 12$$
$$x - y \le 4$$
$$y < 3$$
$$x, y \ge 0$$

To enter the inequalities into the calculator, each one must be solved for *y*. Rewrite the first two constraints into calculator ready form.

To begin, turn on the Inequality Graphing feature by pressing the APPS and choosing **Inequalz**. Press the Y= key to enter the first inequality. When the cursor is placed on the equal sign, inequality options will appear at the bottom of the screen. To enter the first inequality, select ≤ ("less than or equal to") by pressing ALPHA F3 (the ZOOM key) and enter the inequality. To view the graph, press ZOOM key and select **6:ZStandard**. The Y= and graph screens are shown below.





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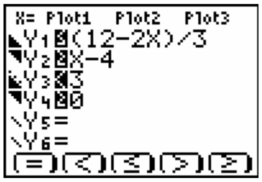
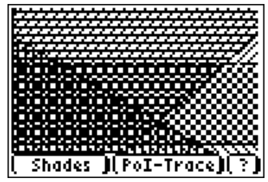
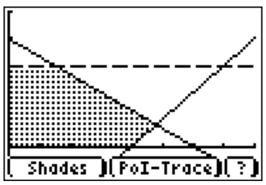
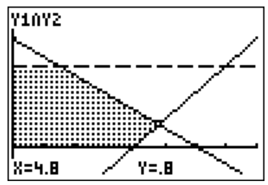
Enter the other three inequalities written in terms of *y*. The remaining condition will be entered by limiting the viewing window – set the window to *x*[0, 8] and *y*[–1, 5], then press [GRAPH]. The Y= and graph screens are shown below.





The screen is full of shadings that are difficult to distinguish from each other. Press [ALPHA] [F1] and choose **1: Ineq Intersection** to see the intersection of the shaded regions (as shown below). Press [ALPHA] [F4] to find the points of intersection, which are (4.8, 8), (1.5, 3), and (4, 0). Note that the two corner points on the *y*-axis will not be indicated because the constraint $x \geq 0$ was never entered.





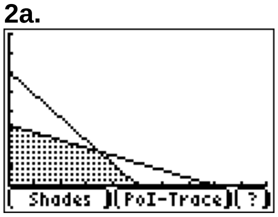
Discuss With Students Answers:

1. *$y \leq (12 - 2x)/3$; $y \geq x - 4$*

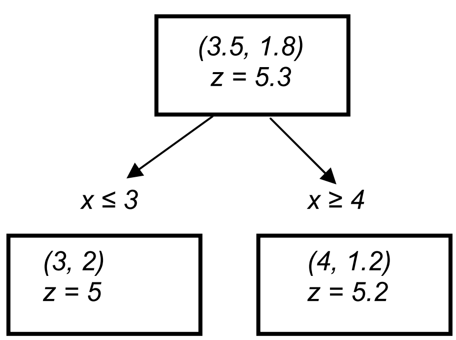
Student Page Answers:

1. *Maximize $z = x + y$; Constraints: $2x + 5y \leq 16$, $6x + 5y \leq 30$; $x, y \geq 0$*

2a.



2b. *(0, 0), (5, 0) (3.5, 1.8), (0, 3.2)* **2c.** *optimal solution is $z = 5.3$ when $x = 3.5$ and $y = 1.8$* **3.** *The values of x and y are not integers.* **4.** *$z = 31/6$; $x = 25/6$, $y = 1$* **5.** *$z = 5$, $x = 3$, $y = 2$* **6.** *$z = 5$ See the chart below.*



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Name ____ Date _______________

NUMB3RS Activity: Branch and Bound

Charlie and Amita are trying to determine where an FBI mole will make the next information drop. Charlie proposes that "by running an algorithm that analyzes the features of the existing meeting places to find key variables, then applying a branching and bounding algorithm, I can hopefully give you a location for the mole's next probable destination."

Charlie is talking about a program to optimize the likelihood of an outcome. One method of optimization often studied in high school math courses is linear programming. The branch and bound algorithm Charlie mentions can be used when integer solutions are desired, but not attained.

Consider the following hypothetical situation: Charlie is breaking codes for the FBI using two different methods: *brute force* and *linear cryptanalysis*. To break a code with the first method, Charlie needs 2 hours of analysis and 6 hours of runtime on the computer. The second method requires 5 hours of analysis and 5 hours of runtime on the computer. If he has 16 hours of analysis and 30 computer hours available, what is the greatest number of codes that Charlie can break?

1. Write the goal and constraints for the problem. Let x represent the number of codes broken with brute force and let y represent the number of cases broken with linear cryptanalysis.

To solve a linear programming problem, first graph the system of inequalities given by the constraints. The resulting area is called the feasible region and the corner points of the region are the potential optimal values. The goal of the problem is to either maximize or minimize an expression; the corner point with the coordinates that satisfy that goal provides the optimal solution.

2. **a.** Use the Inequality Graphing App on your calculator to graph the feasible region. Set the viewing window to x[0, 10] and y[–1, 8].

b. Identify the corner points of the feasible region.

c. Evaluate the goal expression for each corner point and determine the optimal solution.

3. Partially breaking a code is not a desirable result. What is wrong with your solution to Question 2c?

The branch and bound algorithm will allow us to force the variables to be integers, while still optimizing the situation. The first step is to *branch* on a variable that is not an integer, creating two new problems. For simplicity with the calculator, we will choose to branch on y. The current solution is $y = 1.8$, so one branch adds the constraint $y \le 1$ to the original problem and the other branch adds $y \ge 2$.

4. Solve the system again with the added constraint $y \le 1$.

5. Solve the system with the added constraint $y \ge 2$.

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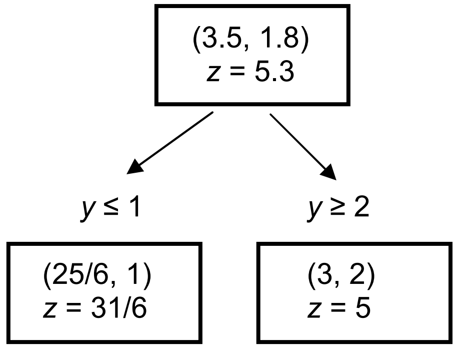
The solutions are shown in the chart below.



The right branch is complete, since the solution is integral. The left branch appears to require more branching. However, rounding down shows that the solution to this branch will be no better than 5. We previously determined an integral solution with that value from the right branch; so the left branch is *bounded* by that value and our optimal solution is 5. This means that Charlie can break at most 5 codes.

6. Repeat the branch and bound algorithm branching on the x variable. Make a chart to show progress.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

When more than two variables are used in a linear programming problem, the graphical analysis becomes more challenging. For these cases, an approach called the Simplex Method can be used. Investigate this method, starting with the web Site below.

http://people.hofstra.edu/faculty/Stefan_Waner/RealWorld/tutorialsf4/frames4_3.html

Additional Resources

- The Web site below shows examples of the simplex method and dual simplex algorithm.
  http://www.egwald.com/operationsresearch/orpage.php3

- The Web site below uses Linear programming with Major League Baseball.
  http://riot.ieor.berkeley.edu/~baseball/

- The Web site below has a Flash Application of the simplex method as well as an add-in to Excel that provides instruction for three different algorithms for solving linear programming. The first two are simplex methods that traverse the boundary of the feasible polytope and the last is an interior point method.
  http://www.me.utexas.edu/~jensen/ORMM/methods/unit/linear/index.html

- The Inequality Graphing App comes preloaded on the TI-84 Plus and TI-84 Plus Silver Edition graphing calculators. For the TI-83 Plus or TI-83 Plus Silver Edition, this App can be downloaded for free from **http://education.ti.com/inequality**.

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