$\dfrac{\sin(x)}{x}$

by Dave Slomer

The function $y = \dfrac{\sin(x)}{x}$ serves mathematics (and, therefore, mathematics students) well. Since this activity is all about it, giving it a name will make it easier to communicate about it. Let's define it to be function ***Q***. In fact, you might want to **Define Q(x)=sin(x)/x** on the home screen and maybe **Zoom Decimal** on the graph screen before proceeding.

1. What is the domain of *Q*?
2. Is *Q even*, *odd*, or neither? Why?
3. What is a formula for all of its *x*-intercepts?
4. Is 0 an *x*-intercept? Why? Does your formula for the *x*-intercepts reflect this?
5. Note that $\dfrac{\sin(x)}{x} = \dfrac{1}{x} \cdot \sin(x)$. Graph *Q* in the **Zoom Decimal** window and tell why this function is sometimes said to have a "shrinking" or "variable" amplitude.
6. Approximate the range of *Q*. To get as accurate a range as possible, you might want to use the graph screen $\boxed{\text{F5}}$ (**Math**) $\boxed{3}$ (**Minimum**) and/or $\boxed{4}$ (**Maximum**) commands.
7. Find all local extreme points of *Q* in $[0,8\pi]$. While there is no nice formula (such as exists for the *x*-intercepts) for the *x*-coordinates of the extreme points, the <u>differences</u> of the *x*-coordinates of consecutive extreme points might have a sort of niceness to it. Explore.
8. **Zoom In** at $x = 0$ and/or look at a table of values for x near 0. Use this information to approximate $\lim_{x \to 0} \dfrac{\sin(x)}{x}$.
9. Use the home screen $\boxed{\text{F3}}$ (**Calculus**) $\boxed{3}$ (**limit**) command to have your TI-89 confirm that you are correct about your approximation for $\lim_{x \to 0} \dfrac{\sin(x)}{x}$. The command you will give is **limit(sin(x)/x,x,0)**.
10. There are many reasons why $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$.

- L'Hopital's Rule seems to provide a short, simple "proof" of this fact, but the very limit that we want to compute is needed for the proof that the derivative of sin(*x*) is cos(*x*)! At least, though, L'Hopital provides a way to never forget the limit! So, use L'Hopital's Rule to compute $\lim_{x \to 0} \dfrac{\sin(x)}{x}$.
- Since this limit is such an important fact of mathematical life, and since there are a few interesting geometric proofs based on the "Squeeze Theorem", you might want to look for one or two. Can you understand one of the proofs well enough to reproduce it?

11. Find $\lim_{x\to\infty} \frac{\sin(x)}{x}$. Zoom to some window (or table) in which you feel confident enough about the graph's behavior that you can predict this limit with confidence.

12. Let your '89 compute the limit. Does it agree with you?

13. Why can you not use L'Hopital to compute this limit?

14. But why is the limit 0? Use the "Squeeze Theorem" to prove it. The functions you need to "surround" $q(x) = \frac{\sin(x)}{x}$ are $1/x$ and $-1/x$, as shown in figure 1 in the **Zoom Decimal** window (with the axes turned off for more clarity). You will need to first establish that those two rational functions actually do surround Q. Start with the 3-part inequality that defines the range of the sine function, then do a little algebra. It will take 2 cases—one for positive x, one for negative x.

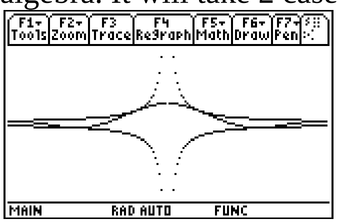


Fig. 1

15. Recall the definition of *horizontal asymptote*: "The line $y = L$ is a horizontal asymptote for a function f if and only if $\lim_{x\to\infty} f(x) = L$." What is the equation of the horizontal asymptote of $q(x) = \frac{\sin(x)}{x}$?

16. How many times does $q(x) = \frac{\sin(x)}{x}$ cross its horizontal asymptote?

17. True or false: *A function may never cross its horizontal asymptote.* Why? (Give a counterexample.)

18. True or false: *A function may never cross its vertical asymptote.* Why?

19. Clearly Q is not continuous at $x = 0$. Explain why, using the <u>formal</u> definition of continuity.

20. The type of discontinuity that Q has at $x = 0$ is called often called a *removable discontinuity*", but is nothing more than a hole. What are the coordinates of the hole? Remove the hole and make Q continuous at $x = 0$ by defining Q(0) to have the proper value.

21. Discuss $Q'(0)$, $\lim_{x \to 0^+} Q'(x)$, and $\lim_{x \to 0^-} Q'(x)$.

22. Recall the definition of derivative (as a limit of a difference quotient). The limit $\lim_{x\to 0} \frac{\sin(x)}{x}$ represents the derivative of some function at some x. What function at what x?

23. Zoom in on that function at that x to estimate the derivative. This gives you one more reason to "know" what $\lim_{x\to 0} \frac{\sin(x)}{x}$ equals.

24. Do you think the improper integral $\int_1^\infty \frac{\sin(x)}{x}\,dx$ converges or diverges? Why?

25. Neither you nor your ’89 can find a “closed-form” anti derivative of Q. Use the Fundamental Theorem of Calculus to produce an “area function” of the form $\int_a^x f(t)dt$ whose derivative is Q (and thus is an antiderivative). Make $Q(1) = 0$. Graph the antiderivative in the **Zoom Decimal** window. Take the derivative of the area function to make sure its derivative is Q.

26. Taylor series exist for Q about $x = 0$. To see that this is so, give the home screen command $\boxed{\text{F3}}$ (**Calculus**) $\boxed{9}$ (**Taylor**) command to get several Taylor polynomials (of varying orders, maybe up to 7^{th} degree) for $\sin(x)$ about $x = 0$. Use the command **Taylor(sin(x),x,3)**, for example, to get the 3^{rd} degree Taylor polynomial for $\sin(x)$ about $x = 0$.
(Note: Taylor series about $x = 0$ are called Maclaurin series, which are <u>always</u> expanded about $x = 0$. To abbreviate below, the phrase “Maclaurin series” will be used occasionally instead of the much longer “Taylor series about $x = 0$.”)

(a) Note that every term in every Maclaurin polynomial for the sine has an x in it, so an x can be always be factored out. Do so. Then divide by x. What did you get?
(b) Could this be the Maclaurin series for Q?
(c) Is dividing by x <u>legal</u> if we are looking for Maclaurin polynomials for Q?
(d) It may be hard to believe that a Taylor series about $x = 0$ for Q even exists. Find $Q'(x)$ and $Q''(x)$. [You could do it by hand, but, looking ahead, be sure you know how to find, say, the 4^{th} derivative of Q <u>by calculator</u>: ***d*****(sin(x)/x,x,4)**.]
(e) What are $Q'(0)$ and $Q''(0)$?
(f) Using the important relationship between differentiability and continuity that says “***differentiability implies continuity***”, explain why $Q'(0)$ is undefined.
(g) Use the quotient rule for derivatives as the basis for your explanation about why $Q^{(n)}(0)$ [the n^{th} derivative of Q at 0] is undefined for all n.
(h) The discontinuity of Q at 0 is removable by applying the definition of continuity. Define a <u>new</u>, *piecewise* function, R, to be a “continuous version” of Q.
(i) Finding the 1^{st} degree Taylor polynomial for Q about $x = 0$ amounts, in essence, to finding it for R. [Since R is continuous at 0, it’s perfectly legal.] Find the <u>**6th**</u> degree Taylor polynomial for R about $x = 0$. Don’t be impatient.
(j) What do you think took so long? How do you think, if $Q^{(n)}(0)$ is always undefined, there could be a Taylor polynomial about $x = 0$?
(k) Use your ’89 to find the 1^{st}, 2^{nd}, 3^{rd}, and 4^{th} derivatives of $Q(x)$ at 0 (see figure 2).

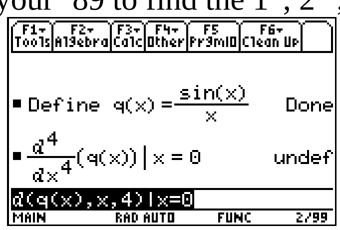


Fig. 2

(l) Use your ’89 to find the **limit** of the 1st, 2nd, 3rd, and 4th derivatives of $Q(x)$ as $x \to 0$ (see figure 3).

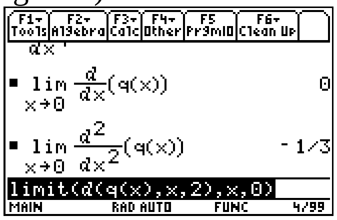
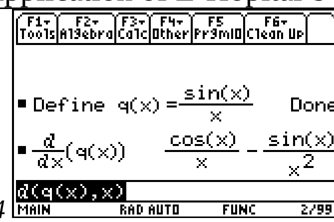
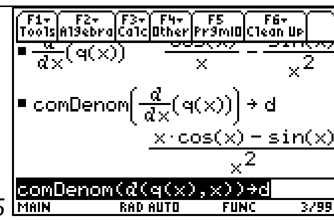
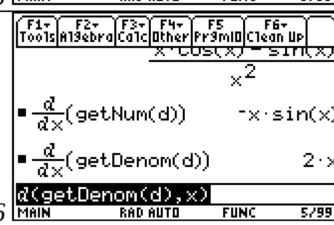
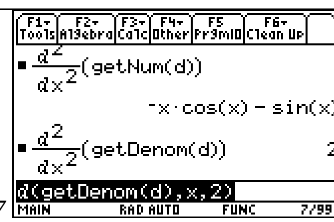


Fig. 3

(m) Use L’Hopital’s Rule to find the limit of $Q'''(x)$ and $Q''''(x)$ as approaches 0. You could do it by hand, but try the procedure outlined below for these and also $Q^{(5)}(0)$ and $Q^{(6)}(0)$. Be sure that you have “0/0” before each application of L’Hopital’s Rule.

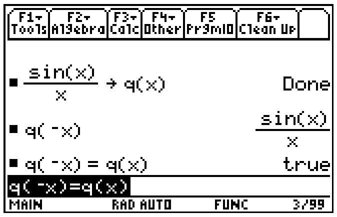
The first derivative of Q isn’t of the right form for L’Hopital’s Rule—it’s not even a quotient, let alone of the form “0/0”.	 *Fig. 4*
The $\boxed{\text{F2}}$ (**Algebra**) $\boxed{6}$ (**comDenom**) command combines terms that are fractions into one term that is a quotient, which L’Hopital can help with. Store the quotient into a variable, such as **d**, to make life easier later.	 *Fig. 5*
L’Hopital requires the derivative of numerator and denominator. You can pull the quotient apart by using the $\boxed{\text{F2}}$ (**Algebra**) $\boxed{\text{B}}$ (**Extract**) $[\blacktriangleright]$ $\boxed{1}$ (**getNum**) and $\boxed{2}$ (**getDenom**) commands. You can take the derivative of each at the same time, checking to see if both are still 0 or not.	 *Fig. 6*
They are both still 0, so repeat the process, but this is equivalent to taking the second derivative of the original numerator and denominator of the quotient stored previously into **d**. The numerator is still 0, but the denominator is not, so the result is 0/2, which is 0.	 *Fig. 7*
Had both numerator and denominator still been 0, the commands ***d*(getNum(d),x,3)** and ***d*(getDenom(d),x,3)** would amount to the third application of L’Hopital. (etc.)	

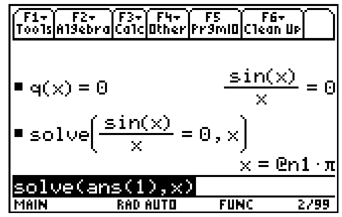
(n) Plug those values into the Maclaurin series formula (stated below) and what do you get? $f(x) = f(0) + f'(0)\cdot x + \dfrac{f''(0)\cdot x^2}{2!} + \dfrac{f'''(0)\cdot x^3}{3!} + \ldots + \dfrac{f^{(n)}(0)\cdot x^n}{2!} + \ldots$

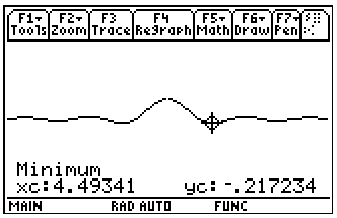
(o) How does the 6th degree Maclaurin polynomial for Q compare with what you got earlier?

(p) Graph Q with the 6^{th} degree Maclaurin polynomial for Q. How would you say the Mac fits Q near 0?

Selected answers, hints, suggestions, comments, etc.

2. 

3. 

6. 

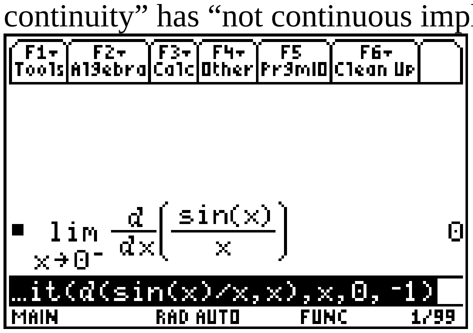
7.

x @ extremum	4.49341	7.72525	10.9041	14.0662	17.2208	20.3713
Distance between consecutive x's		3.23184	3.17885	3.1621	3.1546	3.1505

Find the max and min in the window [500π,502π] by [-.001,.001]. Find their difference. Interesting? Predictable? Provable?

19. Even though $\lim_{x\to 0}\frac{\sin(x)}{x}$=1, Q is not defined at 0.

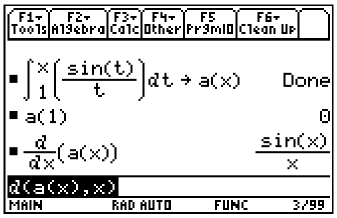
21. Since Q is not continuous at 0, Q' (0) undefined, because "differentiability implies continuity" has "not continuous implies not differentiable" as its *contrapositive*.



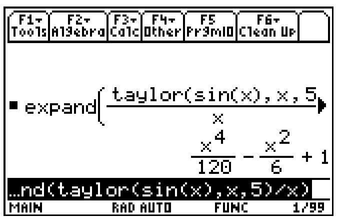
Make the last argument *positive* **1** to get the right-hand limit.

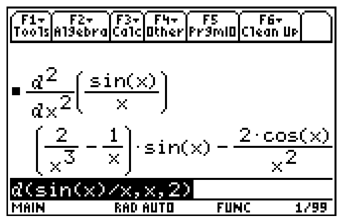
22. The derivative of $f(x)$ = sin(x) at (0,0) is defined as $\lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h}=\lim_{h\to 0}\frac{\sin(h)}{h}$, so that's what this limit represents, derivative-wise. Whether it's x or h approaching 0, the limits say identical things. (Also, that derivative could be defined as $\lim_{x\to 0}\frac{\sin(x)-\sin(0)}{x-0}=\lim_{x\to 0}\frac{\sin(x)}{x}$.)

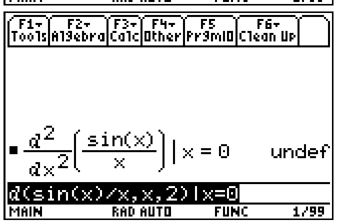
24. Books on Advanced Calculus confirm that the integral converges (is finite), but how to compute it isn't clear. Derive™ announces "Dubious accuracy" and reports 1.5676 when asked for 5 digits of precision. It does the same and then works for quite awhile before returning 1.124909278 when asked for 10 digits of precision. It isn't hard to believe that the area is finite when you consider that the graph is indistinguishable from the *x*-axis throughout most of the **Zoom Standard** window. But of course that <u>proves</u> nothing.

25. 

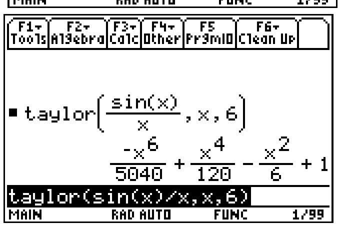
26.

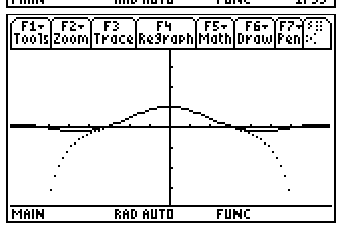
(a) 

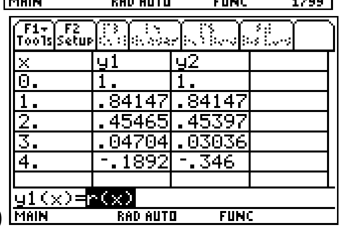
(d) 

(e) 

(h) 

(i) 

(o) 

(p) 

x	y1	y2
0.	1.	1.
1.	.84147	.84147
2.	.45465	.45397
3.	.04704	.03036
4.	-.1892	-.346

Not bad up to about 3. Error analysis required for good response.