Squares and Cubes

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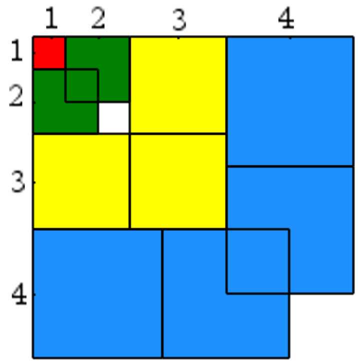
Student Activity

7 8 9 10 **11** **12**

TI-Nspire | Investigation | Student | 30 min

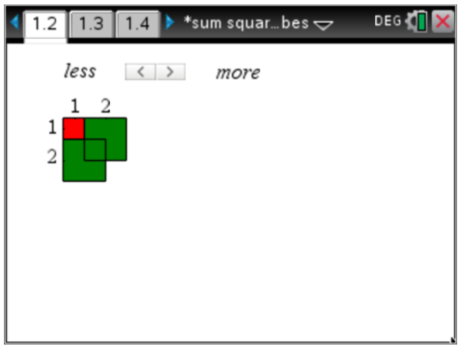
Introduction



How much mathematics can we see in a diagram?

The diagram shown opposite is a powerful example of the visual representation of a mathematical relationship. The diagram shows a series of squares with successively increasing area and frequency. Notice that in some cases there is a small overlap between the squares and that this overlap would exactly fill the corresponding empty space. An animated version of this diagram is available on the TI-Nspire file. Study the diagram and then answer the questions.

Visual Representation



Open the TI-Nspire document "Squares and Cubes".

Navigate to page: 1.2 and use the **show** slider to reveal more or less of the image.

Think about the different ways of expressing the total area of this shape. In the image shown opposite the 2 x 2 squares overlap. The total area of the shape is the combination of the 1 x 1 square and two of the 2 x 2 squares for a total area:

$$1 + 4 + 4 = 9 \text{ units}^2.$$

Question: 1.

Change the slider to reveal more of the shape making the third set of squares visible; then answer the following questions:

a) Imagine you had some red, green and yellow paper and had to cut out the corresponding squares. Describe the colour, size and quantity of each group of squares.
Red: 1 x 1 (Qty 1) + Green: 2 x 2 (Qty 2) + Yellow: 3 x 3 (Qty 3)

b) Calculate the total area of the squares in part (a) by adding up all the component shapes. (Show working)
1 x 1 x 1 + 2 x 2 x 2 + 3 x 3 x 3 = 36

c) Using the overall dimensions of the shape to calculate the total area. (Show working)
$(1 + 2 + 3)^2 = 36$

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Question: 2.

Change the slider to reveal more of the shape making the fourth set of squares visible; then answer the following questions:

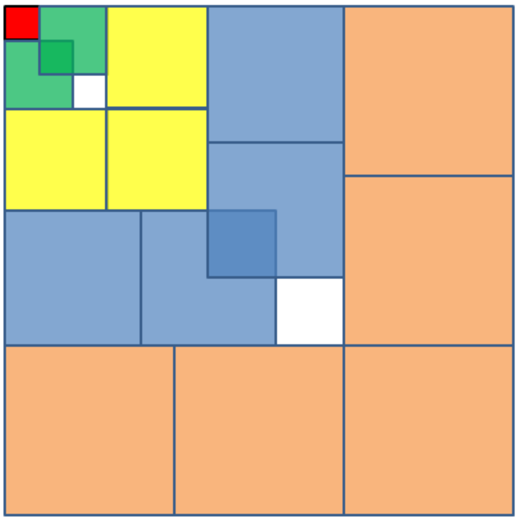
a) Imagine you had some red, green, yellow and blue paper and had to cut out the corresponding squares. Describe the colour, size and quantity of each group of squares.
Red: 1 x 1 (Qty 1) + Green: 2 x 2 (Qty 2) + Yellow: 3 x 3 (Qty 3) + Blue: 4 x 4 (Qty 4)

b) Calculate the total area of the squares in part (a) by adding up all the component shapes. (Show working)
$1 \times 1 \times 1 + 2 \times 2 \times 2 + 3 \times 3 \times 3 + 4 \times 4 \times 4 = 100$

c) Using the overall dimensions of the shape to calculate the total area. (Show working)
$(1 + 2 + 3 + 4)^2 = 100$

Question: 3.

Draw a diagram of shape 5, the next one in the sequence, and determine the total area using both techniques.



Total Area: $1 \times 1^2 + 2 \times 2^2 + 3 \times 3^2 + 4 \times 4^2 + 5 \times 5^2$
$= 225$

Total Area: $(1 + 2 + 3 + 4 + 5)^2$
$= 225$

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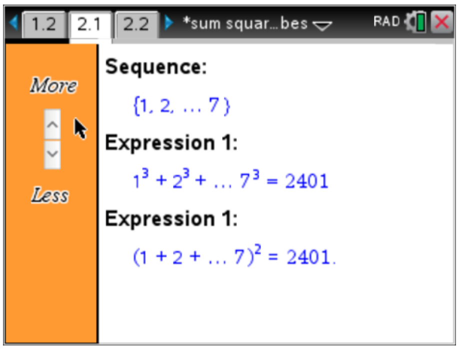
Numerical Representation

Navigate to page 2.1 and use the slider on the left hand side to adjust the value of n. Observe the two expressions as *n* is changed.



Question: 4.

Explain how the two mathematical expressions relate to the original diagram.

$1^3 + 2^3 + \ldots n^3 = 1 \times 1^2 + 2 \times 2^2 \ldots n \times n^2$, where the squared term is the area and the multiplier is the quantity of squares. This formula relates to the sum of the individual squares.

$(1 + 2 + \ldots n)^2$, the expression inside the parenthesis is the total width (and height) of the overall square. Overlaps for the 'even' individual squares overlap by the same amount as the missing section, therefore the area of the overall square is $(1 + 2 + \ldots n)^2$. As both calculations represent the same area, these two expressions must be equal.

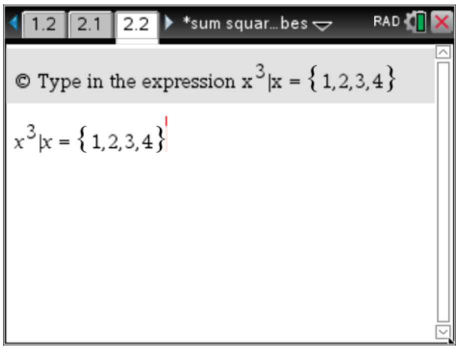
Navigate to page 2.2, this is a calculator application.

A sequence of numbers can be generated by the rule:

$$x^3 \text{ where } x \in \{1,2,3 \ldots n\}$$

To see how this 'rule' works, enter the rule in the calculator (shown opposite) including the substitution for the number set:

{1, 2, 3, 4}



The '|' symbol can be found by pressing [**Ctrl**] + [**=**]

Question: 5.

Write down the set of numbers produced. {1, 8, 27, 64}

Question: 6.

Calculate the sum of these numbers. 100

Hint: Type **sum(** then copy and paste the previous answer.

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Another way to produce a set of numbers and add them in a single step is to use the mathematical summation tool, sigma.

$$\sum_{start}^{finish} rule$$

Navigate to page 3.1

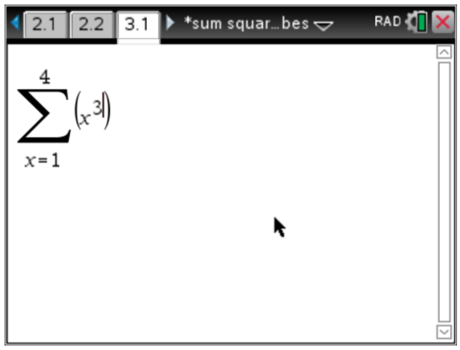
The summation template can be found in the template fly-out or by selecting 'sum' from the calculus menu. The rule being used in this example has the variable x. So the sum of the first 4 integers cubed would look like this:

Start: $x = 1$ -- *initial value used by the rule*

Finish: 4 -- *final value used by the rule*

Rule: x^3 -- *rule used to generate the numbers*

Expression: $\sum_{x=1}^{4} x^3$



Question: 7.

Determine the sum of the first 5 numbers cubed.

Include the sigma notation and expression in the answer.

$$\sum_{x=1}^{5} x^3 = 225$$

Question: 8.

Determine the sum of the first 5 numbers and then square the result.

Include the sigma notation and expression, note that appropriate placement of the squared sign will allow for this calculation to be executed in a single line.

$$\left(\sum_{x=1}^{5} x\right)^2 = 225$$

Question: 9.

Determine the sum of the first 10 cubed integers and compare this with the sum of the first 10 integers and square the result.

$$\sum_{x=1}^{10} x^3 = \left(\sum_{x=1}^{10} x\right)^2 = 3025$$

Question: 10.

Write down an algebraic rule from the sigma notation: $\sum_{n=1}^{x} n^3$ and compare this with $\sum_{n=1}^{x} n$

$\sum_{n=1}^{x} n^3 = \frac{x^2(x+1)^2}{4}$ and $\sum_{n=1}^{x} n = \frac{x(x+1)}{2}$ therefore $\left(\sum_{n=1}^{x} n\right)^2 = \frac{x^2(x+1)^2}{4}$ same as sum of cubes.

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Question: 11.

Use induction to prove that $(1+2+3...n)^2 = 1^3+2^3+3^3...n^3$

Step 1 – Show true for n = 1

Step 2 – Assume true for n (ie rule above)

Step 3 – Show true for n + 1

[Hint: It may be useful to know the rule for the sum of the first n integers.]

Show true for n = 1: $1^3 = 1^2$

Assume true for n: $(1+2+3...n)^2 = 1^3+2^3+3^3...n^3$

Show true for n + 1:

$$\left(1+2+...n+n+1\right)^2 = 1^3+2^3+...n^3+(n+1)^3$$

Let $s = 1+2+...n$

$$\left(s+n+1\right)^2 = (1^3+2^3+...n^3)+(n+1)^3$$

Note that $s^2 = (1^3+2^3+...n^3)$

$$s^2+2sn+2s+n^2+2n+1 = s^2+n^3+3n^2+3n+1$$

$$2sn+2s = n^3+2n^2+n$$

$$2s(n+1) = n(n^2+2n+1)$$

$$2s(n+1) = n(n+1)^2$$

$$s = \frac{n(n+1)}{2}$$

Formula for sum of first n integers

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