

UNDERSTANDING LIMITS

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(Please feel free to email me questions and /or comments.)

Key Topic: Limits and Continuity

Abstract:

This activity is designed to show students how to use a TI-89 to find a limit by graphing, investigating a table, and by using the $\boxed{F3}$ calc menu. After investigating the limits of some continuous and non-continuous functions, students will be given an example for which these three methods don't, at first, appear to give the same results. Their assignment is to figure out why.

The first 3 examples, although each can be done independently by a student, are best done as a classroom or group activity. Ample directions are given on how to use the TI-89, and the 6 multipart questions accompanying these examples are designed to stimulate a student's understanding of the concept of a limit. In fact, it is expected (and perhaps desired) that students who have just been introduced to the concept of a limit will give incorrect answers; thus initiating a dialogue on the meaning of a limit.

After investigating these 3 examples, students will be given more examples and questions designed to reinforce their understanding of limits. These can be done independently, or in small groups, or be given as an assignment. The last exercise is the example of a limit for which these three methods don't, at first, appear to give the same results. Figuring out why this happens will require the student to have not only an understanding of limits, but also an understanding of using a graphics calculator such as the TI-89.

The **ANSWER** section of this activity is designed for teachers. It gives possible answers which students may give, and the reason they may have given them.

Prerequisite Skills:

- Ability to graph functions on a TI-89.
- A basic, introductory understanding of the concept of a limit.
- Ability to find the domain of a function.

Degree of Difficulty: Easy to moderate

Needed Materials: TI-89

NCTM Principles and Standards:

- Content Standards – Algebra
 - Represent and analyze mathematical situations and structures using algebraic symbols
 - Use mathematical models to represent and understand quantitative relationships
 - Draw a reasonable conclusion about situation being modeled

- Process Standards
 - Representation
 - Connections
 - Problem Solving

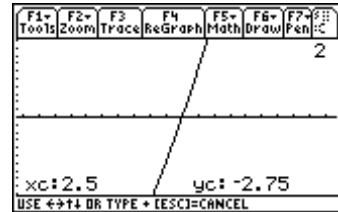
UNDERSTANDING LIMITS

If you understand graphs of functions, you won't have any trouble understanding limits. To show you what I mean consider the expression $\lim_{x \rightarrow 3} f(x)$. The question that is being asked by this expression is: What happens to the values of $f(x)$ as x approaches 3 from both the left and the right? It *is not* asking for the value of $f(3)$. Sometimes the value of $f(3)$ and $\lim_{x \rightarrow 3} f(x)$ are the same, but quite often they are not. Our first example shows you a case where they are the same, the other two examples show cases where $f(3)$ and $\lim_{x \rightarrow 3} f(x)$ are not the same.

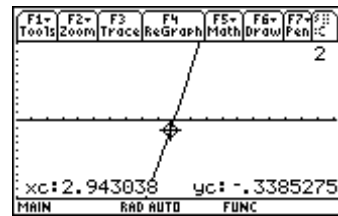
EXAMPLE 1. $\lim_{x \rightarrow 3} x^2 - 9$.

Sketch the graph of $f(x) = x^2 - 9$ in the standard viewing window. ($\text{F2}[6]$). Since we are interested in what happens as x approaches 3, zoom in on this part of the graph. ($\text{F2}[2]$, move the cursor close to 3 on the x -axis, and press ENTER .)

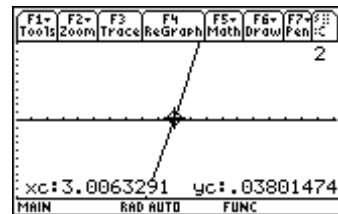
To see what happens as x approaches 3 from both the left and the right, press F3 , trace. Press the left arrow key \leftarrow until the x -coordinate of the cursor is reasonably less than 3. (See the illustration at the right where the x -coordinate of the cursor is 2.5. If you zoomed in from a different location than the one used to produce this illustration, your calculator screen may look different, which is OK.)



Now press the right arrow key \rightarrow one time. You have just taken the first step in finding what happens to the value of $x^2 - 9$ as x approaches 3 from the left. Keep pressing the right arrow key until the x -coordinate is as close to 3 as you can get without exceeding 3. As you do this, pay attention to the y -coordinates. (Again, if your screen does not look exactly the same as the one shown here, it's OK, you just zoomed in from a different location.)



To see what happens as x approaches 3 from the right, press \rightarrow until the x -coordinate is greater than 3, and then press \leftarrow to see what happens when x gets close to 3 without exceeding 3.



QUESTION 1. Based on what you just did, have you found the value of $\lim_{x \rightarrow 3} x^2 - 9$? If so, what is the limit?

We could zoom in again and re-do the trace procedure to get a better idea of what's going on, but this is time consuming. So let's try another procedure: using a table of values instead of trace values.

To do this, press \blacktriangledown [TblSet]. Enter 2.5 in **tblStart**, arrow down and enter 0.1 in Δ **tbl**, press [ENTER], and then press \blacktriangledown [TABLE].

F1-Tools	F2 Setup	F3	F4	F5	F6
x	42				
2.5	-2.75				
2.6	-2.24				
2.7	-1.71				
2.8	-1.16				
2.9	-.59				
x=2.5					
MAIN RAD AUTO FUNC					

Arrow down so that you can see what happens when $x = 3$.

F1-Tools	F2 Setup	F3	F4	F5	F6
x	42				
2.8	-1.16				
2.9	-.59				
3.	0.				
3.1	.61				
3.2	1.24				
x=3.2					
MAIN RAD AUTO FUNC					

QUESTION 2. If $f(x) = x^2 - 9$,

a. What is the value of $f(3)$?

b. What is the value of $\lim_{x \rightarrow 3} x^2 - 9$? Explain how you know this.

c. What is the domain of $f(x)$?

d. Does $\lim_{x \rightarrow 3} x^2 - 9 = f(3)$?

EXAMPLE 2. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.

QUESTION 3. Before using the TI-89 to investigate this limit, answer the following questions about the function $f(x) = \frac{x^2 - 9}{x - 3}$.

a. What is the domain of $f(x)$?

b. Without using a calculator, what do you think the graph of $y = f(x)$ will look like?

c. What do you think is the value of $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$?

QUESTION 4. Graph the function $f(x) = \frac{x^2 - 9}{x - 3}$ on the TI-89, and then answer the following questions.

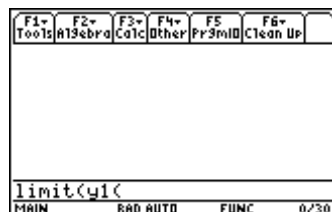
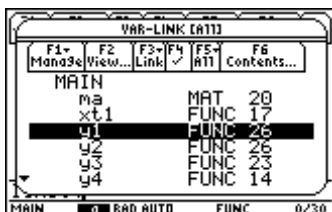
- Is the graph of $f(x) = \frac{x^2 - 9}{x - 3}$ a straight line? Why, or why not?
- Now what do you think is the value of $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$?
- Does $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = f(3)$?

QUESTION 5. Check out your answer by looking at a table of values. In \square [TblSet], enter 2.9 in **tblStart** and 0.01 in Δ **tbl**. Then arrow down until you get to $x = 3$.

F1+	F2+	F3+	F4+	F5+	F6+
Tools	Setup	Table	Other	Pr3mid	Clean Up
x	y2				
2.9	5.9				
2.91	5.91				
2.92	5.92				
2.93	5.93				
2.94	5.94				
x=2.9					
MAIN		RAD AUTO		FUNC	

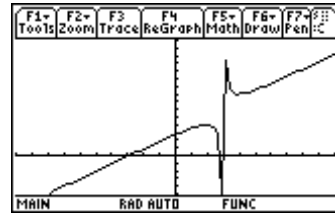
- What is the value of $f(3)$?
- What value does $f(x) = \frac{x^2 - 9}{x - 3}$ approach as x approaches 3?
- In which direction did you just let x approach 3?
- Keep arrowing down to see what value $f(x) = \frac{x^2 - 9}{x - 3}$ approach as x approaches 3 from the opposite direction. What is the value, and what is the direction?

As a final test, let's have the TI-89 find the limit for us. Press \square [HOME] \square [F3] 3. When requested for the function, you could type it in, but why bother, it's already in your calculator. You can retrieve it by pressing \square [2nd] \square [VAR-LINK], arrowing down to **y1**, or wherever you stored the function, and then pressing \square [ENTER]. Complete the limit as shown below in the picture at the right, and then press \square [ENTER].



EXAMPLE 3. $\lim_{x \rightarrow 3} \frac{x^2 - 8}{x - 3}$.

QUESTION 6. Answer the following questions about the function $f(x) = \frac{x^2 - 8}{x - 3}$ whose graph appears at the right.

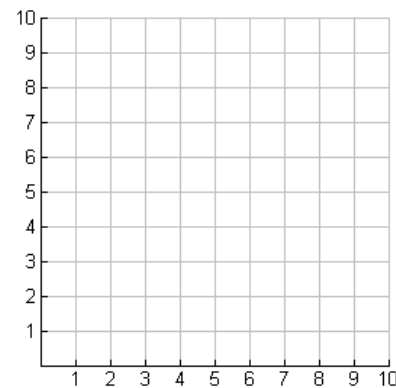


- What is the domain of $f(x)$?
- What is the value of $f(3)$?
- What is the value of $\lim_{x \rightarrow 3} f(x)$? Justify your answer.

EXAMPLE 4. $\lim_{x \rightarrow 3} f(x)$ where $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ 4, & x = 3 \end{cases}$

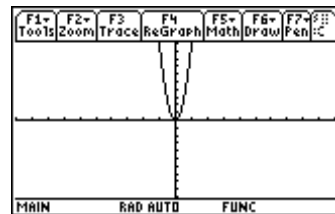
QUESTION 7. Sketch the graph of $y = f(x)$, and then answer the following questions.

- What is the domain of $f(x)$?
- What is the value of $f(3)$?
- What is the value of $\lim_{x \rightarrow 3} f(x)$?



EXAMPLE 5. $\lim_{x \rightarrow 10} f(x)$ where $f(x) = \frac{10x^3 - 100x^2 + 1}{x - 10}$.

QUESTION 8. Pictured at the right is the graph of $y = f(x)$ graphed in the standard viewing window [F2] 6.



Use this as a starting point to investigate the following questions.

- Using only the **Trace** feature of the TI-89, approximate the value of $\lim_{x \rightarrow 10} f(x)$.
- Use a table of values to approximate the value of $\lim_{x \rightarrow 10} f(x)$.
- Use the calc menu ([HOME][F3] 3) to find $\lim_{x \rightarrow 10} f(x)$.

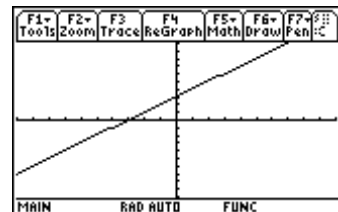
ANSWERS:

1. Answers will vary from 0 to “doesn’t exist” to “no.”
 - The student who jumped ahead and evaluated $x^2 - 9$ when $x = 3$, will answer zero.
 - The student who caught on about the part in the class lecture where the approach from the left has to equal the approach from the right, but doesn’t fully understand the intricacies of evaluating these approaches will answer “does not exist.”
 - The astute student who fully understands the concept of a limit and who read the “based on what you just did” part of the question should answer “no.”

2. a. 0
 - b. 0. Since the graph is smooth (no holes, no gaps), as x approaches 3 from either the left or the right, $f(x)$ will approach 0, the value of $f(3)$.
 - c. All real numbers.
 - d. Yes

3. a. All real numbers except 3.
 - b. Answers will vary.
 - Students having a weak background in algebra will not be able to answer the question.
 - The most likely, but incorrect, reply is that there is an asymptote at $x = 3$.
 - The correct answer is that it is the graph of $y = x + 3$ with a hole in it at $x = 3$.
 - c. Answers will vary.
 - Students having a weak background in algebra may answer “ $\frac{0}{0} = 0$ ” or “ $\frac{0}{0} = 1$.”
 - It is hoped that the students who incorrectly thought there was an asymptote at $x = 3$, would suspect that the limit does not exist.
 - The correct answer is, of course, 6.

4. a. Some students may incorrectly answer yes since $\frac{x^2 - 9}{x - 3} = x + 3$. Those who paid attention to the domain of the function will say it is the straight line $y = x + 3$ with a hole in it at $x = 3$.



- b. 6
- c. No, $f(3)$ is undefined.

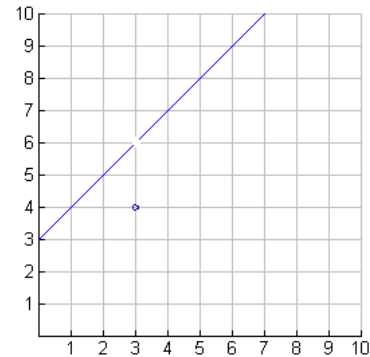
5. a. undefined.
 b. maybe 6.
 c. left.
 d. maybe 6, right.

F1	F2	F3	F4	F5	F6	F7	F8
Tools	Setup	2nd	3rd	4th	5th	6th	7th
x	y2						
2.96	5.96						
2.97	5.97						
2.98	5.98						
2.99	5.99						
3.	undef						
x=3.							
MAIN		RAD AUTO		FUNC			

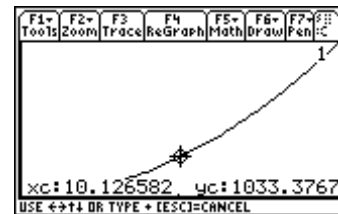
F1	F2	F3	F4	F5	F6	F7	F8
Tools	Setup	2nd	3rd	4th	5th	6th	7th
x	y2						
3.	undef						
3.01	6.01						
3.02	6.02						
3.03	6.03						
3.04	6.04						
x=3.04							
MAIN		RAD AUTO		FUNC			

6. a. $x \neq 3$ b. undefined
 c. Does not exist. The limit from the left does not equal the limit from the right.

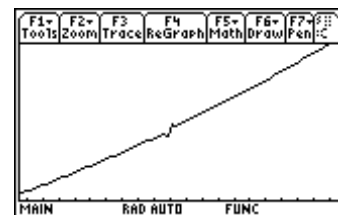
7. a. all real numbers.
 b. 4
 c. 6



8. a. Students should first realize that the limit cannot be found in the standard viewing window. Change to a window with $0 \leq x \leq 20$ and use **ZomFit** (F2) to find an appropriate viewing window. Using trace will show that the limit appears to be between 967 and 1033.



Zooming in on this limit, as pictured at the right, will give new values of a limit between 962 and 1037. Students should recognize that they are not getting closer to a limiting value, but moving away from such. Also, the zig-zag in the graph at $x = 10$ should look suspiciously like an asymptote. Zooming in again will reveal the asymptote. So the limit does not exist.



- b. Using a value of 0.1 in Δx makes it look as though the limit could be approaching 1000. And using a value of 0.01 still makes it look like the limit could be approaching 1000, although students should realize that they are moving away from a limit, not towards a limit. This is confirmed by using a Δx size of 0.001, as pictured below at the right. Here the $f(x)$ values become negative as x approached 10 from the left, but $f(x)$ gets quite large as x approaches from the right. So the limit does not exist.

F1+ Tools	F2 Setup	F3	F4	F5	F6
x	y1				
9.8	955.4				
9.9	970.1				
10.	undef				
10.1	1030.1				
10.2	1045.4				
x=10.2					
MAIN		RAD AUTO		FUNC	

F1+ Tools	F2 Setup	F3	F4	F5	F6
x	y1				
9.98	946.				
9.99	898.				
10.	undef				
10.01	1102.				
10.02	1054.				
x=9.98					
MAIN		RAD AUTO		FUNC	

F1+ Tools	F2 Setup	F3	F4	F5	F6
x	y1				
9.998	499.6				
9.999	-.2				
10.	undef				
10.001	2000.2				
10.002	1500.4				
x=9.998					
MAIN		RAD AUTO		FUNC	

c.

F1+ Tools	F2+ Algebra	F3+ Calc	F4+ Other	F5 Pr3mID	F6+ Clean Up
<ul style="list-style-type: none"> lim y1(x) undef x→10 					
limit(y1(x),x,10)					
MAIN		RAD AUTO		FUNC 1/30	