EXPLORATIONS™

Activity 13

Objectives

- Understand what a slope field represents in terms of $\frac{dy}{dx}$
- Create a slope field for a given differential equation

Materials

- TI-84 Plus / TI-83 Plus
- Graph paper

Introduction to Slope Fields

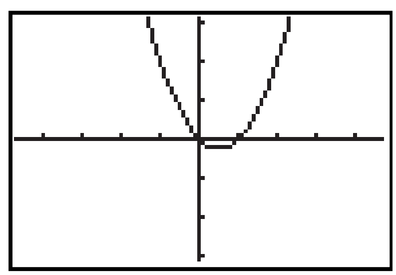
Introduction

One of the ways to visually think of the derivative is to use the idea of *local linearity*. That is, if you look "up close" at the graph of a differentiable function, it appears almost straight, so locally (on a small enough interval) a differentiable function behaves very much like a linear function.

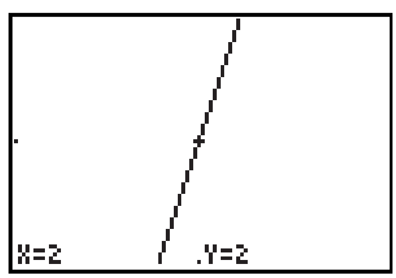
You can experience this directly by using a graphing handheld. If you zoom in repeatedly on a particular point on the graph of a differentiable function, eventually the graph appears indistinguishable from a straight line. The slope of that line is the value of the derivative at that point.

Exploration

Examine the function $y = x^2 - x$ at $x = 2$. First, graph **Y1 = X^2–X** in the **ZDecimal** viewing window.



Now [TRACE] over to the point (2, 2) on the graph, and press [ENTER] to re-center the viewing window at that point. Now, zoom in three or four times until the graph appears straight.



In fact, if you count the pixels you will see that the graph rises 3 pixels for each 1 pixel run, so the slope is about 3. That is an approximation for the derivative

$$\frac{dy}{dx}$$

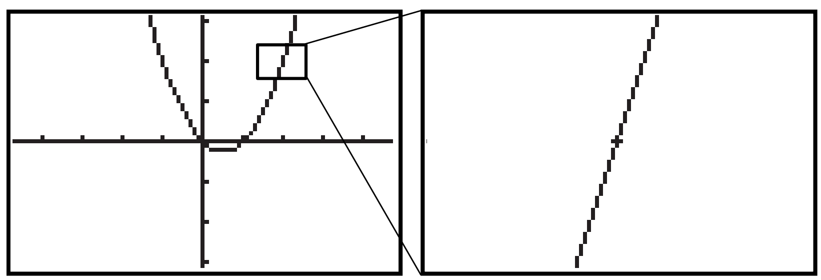
at $x = 2$. The function $y = x^2 - x$ is a relatively simple one, and the actual value of the derivative for any value x is given by the formula

$$\frac{dy}{dx} = 2x - 1$$

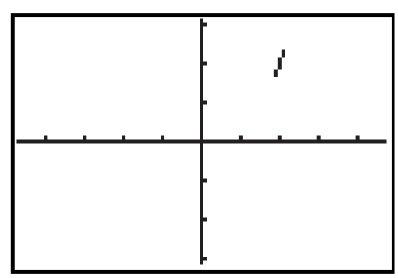
Thus, at $x = 2$,

$$\frac{dy}{dx} = 2(2) - 1 = 3$$

This close-up of the graph represents a magnified view of a small part of the graph in the original viewing window.



Now, imagine that the screen shows only a piece of a graph. This small piece of the graph looks like a line segment centered at the point (2, 2) with a slope of 3.



Suppose that the function was unknown and you were given only the information that

$$\frac{dy}{dx} = 3$$

at the point (2, 2). This small line segment would still be a good local approximation to the graph. Of course, it would be impossible to guess what the rest of the graph would look like without more information.

Find at least two functions other than $y = x^2 - x$ that satisfy the slope condition that

$$\frac{dy}{dx} = 3$$

at the point (2, 2).

A derivative provides *local slope* information about the graph of a function. If the value of the derivative at a particular point is known, then a small piece of the graph of the original function can be visualized at that point. If the derivative value is known at many points, then many small pieces of the original function's graph can be visualized. Viewing several pieces of the graph together could help you visualize what larger pieces of the original function's graph might look like. This is the idea behind what is called a *slope field* (or *direction field*).

Suppose that you are given the derivative of a function, such as

$$\frac{dy}{dx} = f'(x) = 2x - 1$$

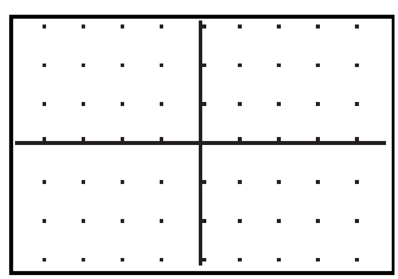
and you want to visualize what the original function's graph looks like. You do not know any actual points on the graph of the function, but the derivative tells you that at any particular point (x_0, y_0), the local slope of the function graph is $2x_0 - 1$. For example, at the origin (0, 0), the local slope is 2(0) – 1 = ⁻1. At the point (⁻3, 1), the local slope is 2(⁻3) – 1 = ⁻7.

If the function's graph goes through the origin, you know that its graph looks approximately like a line segment with slope ⁻1 near the origin. If the function graph goes through the point (⁻3, 1), then its graph looks approximately like a line segment with slope ⁻7 near that point. You could set up a grid (or *lattice*) of sample points that the graph might pass through. At each one of these points, you could draw a small line segment with the appropriate slope using the value of the derivative

$$\frac{dy}{dx}$$

The result would be a *field* of *slope segments* that would visually approximate what the graph of $y = f(x)$ looks like if it goes through any of these points.

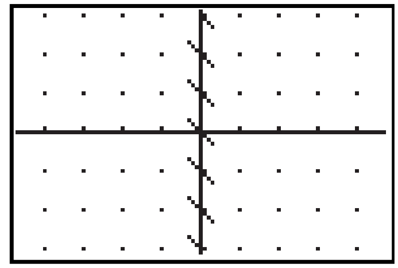
An example of such a grid of sample points would be the points in the **ZDecimal** viewing window with integer coordinates. Produce your own grid by having no functions selected in the **Y=** editor and selecting **GridOn** in the **FORMAT Menu**. Press [GRAPH] to produce the screen shown.



For any point where $x = 0$, the slope indicated by the differential equation

$$\frac{dy}{dx} = 2x - 1$$

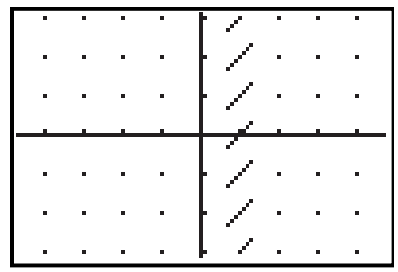
is $2(0) - 1 = {}^-1$, so the grid points along the y-axis should each have a short line segment of slope ${}^-1$, as shown:



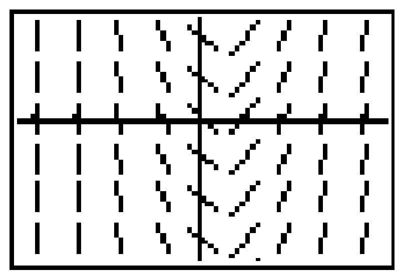
For any point where $x = 1$, the slope indicated by the differential equation

$$\frac{dy}{dx} = 2x - 1$$

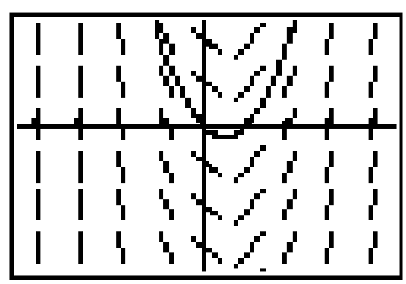
is $2(1) - 1 = 1$, so the grid points along the line $x = 1$ should each have a short line segment of slope 1, as shown.



Continuing in this manner, you can draw in a slope field at all the integral grid points indicated to obtain a slope field like the one shown.



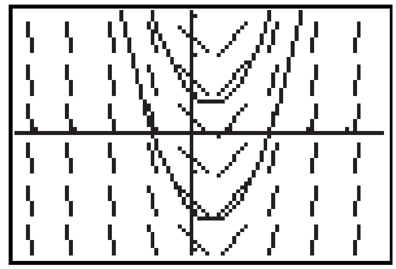
If you sketch one of the solutions to this differential equation, namely $y = x^2 - x$, then the graph should appear to follow the flow of the slope field.



In fact, this screen suggests why the arbitrary constant *C* is important to include when you take an antiderivative. If the graph is shifted vertically up or down, then it still follows the flow of the slope field. Any function of the form $y = x^2 - x + C$ is a solution to the differential equation:

$$\frac{dy}{dx} = 2x - 1$$

Notice how $y = x^2 - x + 1$ and $y = x^2 - x - 2$ also follow the flow of the slope field.



The differential equation

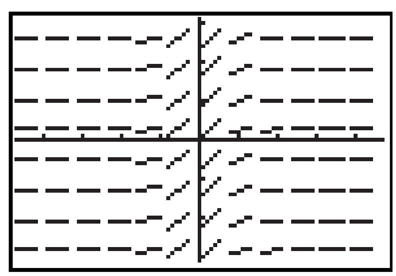
$$\frac{dy}{dx} = 2x - 1$$

is simple enough, and the family of antiderivatives $y = x^2 - x + C$ that are solutions to it are not too hard to figure out. Slope fields are useful in cases in which the exact solutions may not be so easy to find, but the solution curves must be visualized. For example, the differential equation

$$\frac{dy}{dx} = e^{-x^2}$$

is one for which you cannot find a family of solutions in terms of familiar functions such as polynomial, trigonometric, logarithmic, or exponential functions. Nevertheless, you can visualize characteristics of the family of solutions using a slope field.

Does the slope field shown here make sense?



The derivative

$$\frac{dy}{dx} = e^{-x^2}$$

indicates that the slopes should always be positive and close to 1, near $x = 0$, and closer to 0 as you move away from $x = 0$ in either the positive or negative direction.

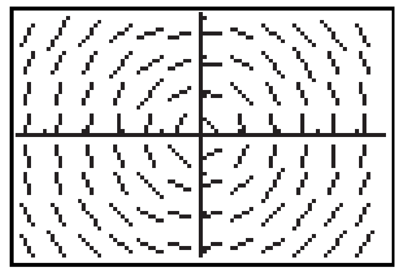
You can also make slope fields for differential equations in terms of *y* or in terms of both *x* and *y*. For example, when you use implicit differentiation on the relation such as the graph of a circle

$$x^2 + y^2 = 9$$

you obtain the differential equation

$$\frac{dy}{dx} = \frac{-x}{y}$$

Shown is a slope field for this differential equation.

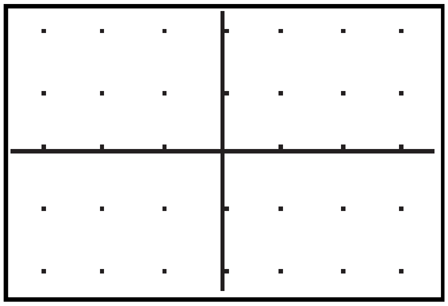


You can see how the family of solution curves for this differential equation appears to be made up of circles centered at the origin. In fact, *any* circle of the form $x^2 + y^2 = C$ leads to the same differential equation.

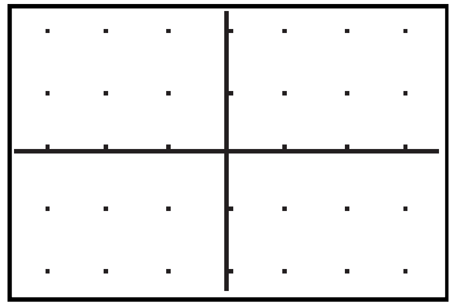
Directions

On the grids provided, draw one line segment at each lattice point to make a slope field that has the appropriate slopes indicated by each differential equation.

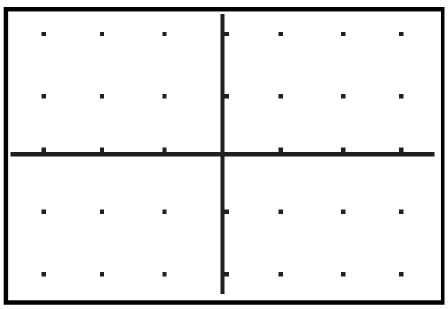
1. $\frac{dy}{dx} = 2 - x$



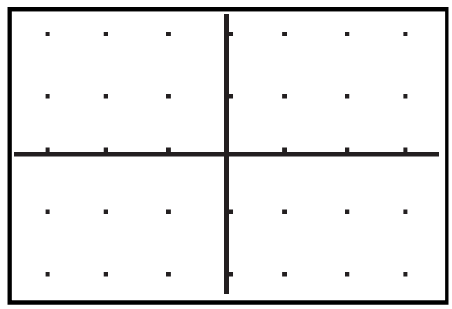
2. $\frac{dy}{dx} = \frac{1}{x}$



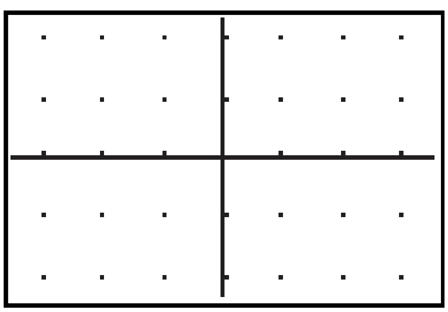
3. $\frac{dy}{dx} = 2^x$



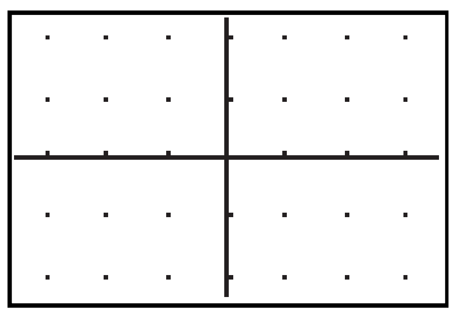
4. $\frac{dy}{dx} = \cos(\pi x)$



5. $\frac{dy}{dx} = y - 1$



6. $\frac{dy}{dx} = \frac{y}{x}$



Given an initial condition for the differential equation of the form

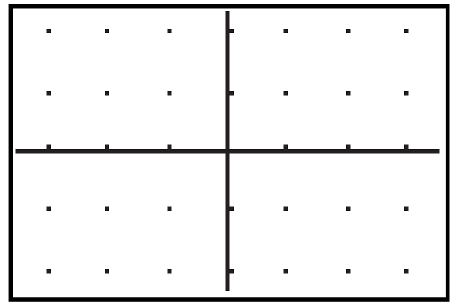
$$F(x_0) = y_0$$

you can sketch an approximate graph of $y = F(x)$ using a slope field. The initial condition tells you that the point (x_0, y_0) lies on the graph of F, and starting at that point, you can sketch a curve extending both left and right that follows the "flow" indicated by the slopes of the line segments in the slope field.

7. For the slope field you made in Question **1**, sketch the graph of a solution

a. $y = f(x)$ satisfying $f(1) = {}^-1$.

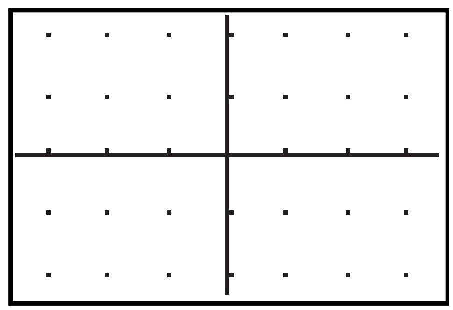
b. $y = g(x)$ satisfying $g({}^-2) = 1$.



8. For the slope field you made in Question **2**, sketch the graph of a solution

a. $y = f(x)$ satisfying $f(1) = 0$.

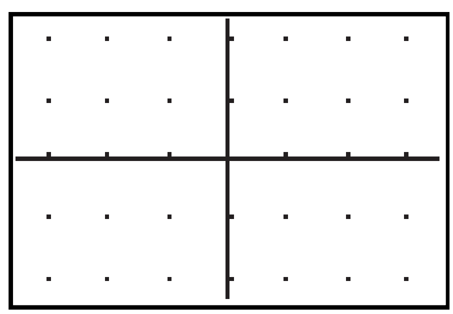
b. $y = g(x)$ satisfying $g({}^-1) = 2$.



9. For the slope field you made in Question **3**, sketch the graph of a solution

a. $y = f(x)$ satisfying $f(0) = 1$.

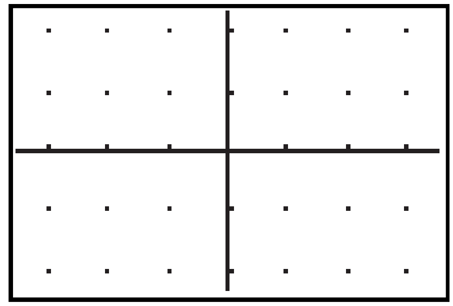
b. $y = g(x)$ satisfying $g(1) = {}^-2$.



10. For the slope field you made in Question **4**, sketch the graph of a solution

a. $y = f(x)$ satisfying $f(1) = {}^{-}1$.

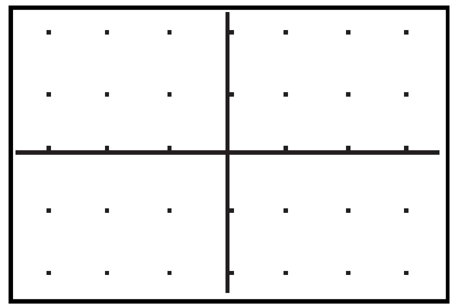
b. $y = g(x)$ satisfying $g({}^{-}2) = 1$.



11. For the slope field you made in Question **5**, sketch the graph of a solution

a. $y = f(x)$ satisfying $f(1) = {}^{-}1$.

b. $y = g(x)$ satisfying $g({}^{-}2) = 0$.



12. For the slope field you made in Question **6**, sketch the graph of a solution

a. $y = f(x)$ satisfying $f(1) = {}^{-}1$.

b. $y = g(x)$ satisfying $g({}^{-}2) = 2$.