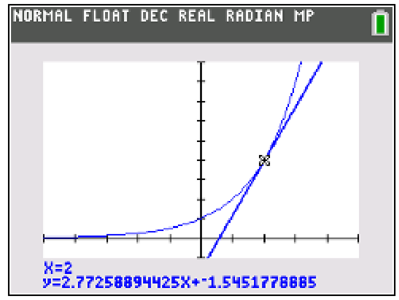


Math Objectives

- Students will graph exponential functions of the form $f(x) = b^x$
- Students will evaluate the exponential function $f(x) = b^x$ for any value of x.
- Students will calculate the doubling time or half-life in a problem involving exponential growth or decay.
- Students will use appropriate technological tools strategically (CCSS Mathematical Practice).

Vocabulary

- Exponential function
- Euler's number (*e*)
- Growth
- Tangent Line
- Decay

About the Lesson

- This lesson involves finding an approximation for the value of the mathematical constant e and apply it to growth and decay problems.

As a result, students will:

- Examine the graphs of multiple exponential functions, comparing their b values.
- Create tangent lines to the exponential function $f(x) = 2^x$, and compare the slopes of the tangent with their points of tangency in an effort to approximate e.
- Use e to aid in real world problems involving population growth, compound interest growth and radioactive decay.

Teacher Preparation and Notes.

This activity is done with the use of the TI-84 family as an aid to the problems.

Activity Materials

- Compatible TI Technologies: TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*

Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

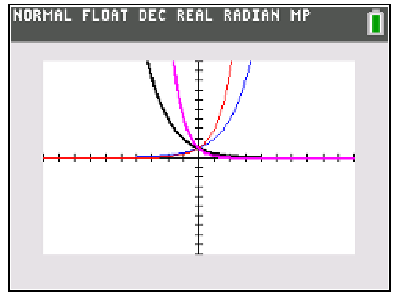
Lesson Files:

Student Activity

Exponential_Growth_84CE_Student.pdf

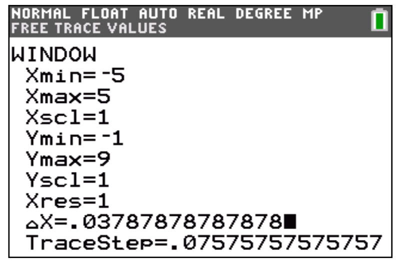
Exponential_Growth_84CE_Student.doc

In this activity, students will find an approximation for the value of the mathematical constant e and to apply it to exponential growth and decay problems. To accomplish this, students are asked to search for the base, b, that defines a function $f(x) = b^x$ with the property that at any point on the graph, the slope of the tangent line (instantaneous rate of change) is equal to $f(x)$. The result is approximating the value of e — Euler's number and the base of the natural logarithms.



Problem 1 – Comparing Growth and Decay Functions

Before beginning this activity, change your window settings to match those to the right.



Enter the function $f(x) = b^x$ with 5 different values of *b* (for *b* > 0). Choose some values that are greater than 1 and some values that are less than one. Then, press **graph** to graph the functions.

Use **trace** to observe how the value of *b* affects the shape of the graph. Use the up and down arrows to move among the curves. Use the left and right arrows to move along the curves.

(a) Write at least three observations about the effect of the value of *b* on the graph of $f(x)$.

Possible Solutions: Answers may vary. Possible observations: graph gets "steeper" as *b* increases and "flatter" as *b* decreases; always passes through the point (0, 1); increasing when *b* > 1 and decreasing when 0 < *b* < 1.

(b) What value of *b* results in a constant function? Explain.

Solution: *b* = 1, this will result in a horizontal line.

(c) Explain why the value of *b* cannot be negative.

Possible Solution: Answers may vary. Possible explanation: Even roots of negative numbers are not real numbers. Consider, for example, $(-1)^{0.5} = \sqrt{-1}$, which is not a real number.

Teacher Tip: In part (c), the problem asks students to explain why the value of b cannot be negative. It may be worthwhile to discuss this in a whole-class setting.

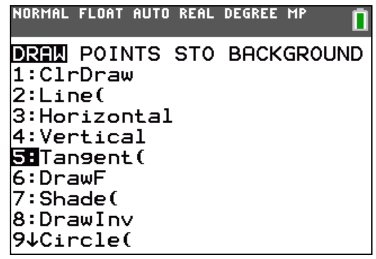
Problem 2 – Finding the Slope of a Tangent Line

Tech Tip: It may be necessary to spend some time reviewing or teaching this skill of creating a tangent line and finding the coordinates of the point of tangency, to make problem 2 run smoother.

Now you are going to graph function $f(x) = b^x$ along with its tangent line. Start by clearing the functions from the **y =** screen. Enter the function $f(x) = 2^x$. Then, press **graph** to view the graph of the function.

Press **2nd prgm** to access the Draw menu. Select **5:Tangent** and press **enter**.

Enter an *x*-value to choose a point where the line will be tangent with the graph of $f(x) = 2^x$. Press **enter**.



The calculator draws the tangent line and displays the equation of the line. Record the *x*-value and the slope of the tangent line.

(a) *x*: ____**answers will vary**____________

(b) slope of tangent: _______**answers will vary**_________

Now find the value of the function $f(x) = 2^x$ at the same point. Press **2nd trace** to open the CALCULATE menu. Select **1:value**. Enter the *x*-value you recorded. Press **enter**.

The calculator displays the *y*-value of the function at this point. This is the value of the function for this value of *x*.

(c) $f(x)$: ____**answers will vary**____________

(d) How does the slope of the tangent line at this point compare to the value of the function, $f(x)$?

Solution: The slope is less than $f(x)$.

Return to the **y =** screen. Change the value of *b* to a nonnegative number of your choice and graph the new function. Draw a tangent line at any point on the graph of $f(x)$.

Record the values of *b*, *x*, $f(x)$, and the slope of the tangent line at *x* in the table below along with your earlier observations.

Sample Table:

b	*x*	*f*(*x*)	slope of tangent at *x*
2	**3**	**8**	**5.545**
3	**1**	**3**	**3.296**
0.5	**0**	**1**	**-0.693**
0.25	**2**	**0.063**	**-0.087**

Return to the **y =** screen and change the value of *b* again. Draw a tangent line for each curve and record your results in the table.

(e) Write at least two observations about the graph and/or the slope of its tangent at *T*.

Possible Solutions: Answers may vary. Possible observations: slope is always positive; as x increases, the slope increases; curve never reaches the x-axis.

Problem 3 – Euler's Number

Slope is a measure of rate of change in a function. In this example, sometimes the slope is ***less than*** *y*, and sometimes it is ***greater than*** *y*. There is only one value of *b* for which the rate of change of the function $y = b^x$ at any point is ***equal to*** the value of the function itself. Can you find an approximate value of this number?

When the rate of change of $y = b^x$ is ***equal to*** the value of the function, the ratio $\frac{\text{slope of tangent at } x}{f(x)}$ will equal one.

Sample Table:

b	$\frac{\text{slope of tangent at } x}{f(x)}$
2	**0.693**
3	**1.099**
0.5	**-0.693**
0.25	**-1.381**

To begin the search for this value of *b*, use the data you have collected to complete the table.

Value of *b* that is closest to 1 and greater than 1: __**Answers will vary. Possible answer: 3**___

Value of *b* that is closest to 1 and less than 1: __**Answers will vary. Possible answer: 2**____

The value of *b* we are looking for must be between these two.

Choose some values of *b* that are between two numbers and repeat the process of graphing the function, drawing a tangent line, recording the value of the function and the slope of the tangent line at that point, and calculating the ratio. Narrow in on the value of *b* that yields a ratio of 1 as closely as you can.

Sample Table:

b	x	$f(x)$	slope of tangent at x	$\frac{\text{slope of tangent at } x}{f(x)}$
2.2	**0**	**1**	**0.788**	**0.788**
2.4	**1**	**2.4**	**2.101**	**0.875**
2.6	**0**	**1**	**0.956**	**0.956**
2.8	**2**	**7.84**	**8.072**	**1.030**
2.7	**0**	**2.7**	**0.993**	**0.993**
2.75	**0**	**2.75**	**1.0116**	**1.0116**

What is this value of b? $b \approx$ ___**2.718**______

Applications

The number you found is an approximation for the mathematical constant *e*. As you discovered, it is unique in that it is the only value of *b* such that $y = b^x$ changes at a rate that is equal to the value of the function itself. It also shows up in a number of functions that model natural phenomena.

Some examples are:

(a) the growth of populations of people, animals, and bacteria;

(b) the value of a bank account in which interest is compounded continuously;

(c) and radioactive decay.

The common feature is that the rate of growth or decay is proportional to the size of the population, account balance, or mass of radioactive material. Growth and decay situations can be modeled by equations of the form $P = P_0e^{kt}$, where P is the current amount or population, P_0 is the initial amount, t is time, and k is a growth constant. An amount is *growing* if $k > 0$ and *declining* if $k < 0$.

The following are examples of exponential growth or decay. For each exercise, write an equation to represent the situation and solve your equation to find the answer.

1. Suppose you invest $1,000 in a CD that is compounded continuously at the rate of 5% annually. (Compounded continuously means that the investment is always growing rather than increasing in discrete steps.) What is the value of this investment after one year?
Two years? Five years?

Solution: Modeling equation: $P = 1000e^{0.05t}$ (where P is the value and t is the time in years);
One year: $1,051.27
Two years: $1,105.17
Five years: $1,284.03

2. A colony of bacteria is growing at a rate of 50% per hour. What is the approximate population of the colony after *one day* if the initial population was 500?

Solution: Modeling equation: $P = 500e^{0.5 \cdot 24}$ (where P is the population)
After one day: about 81,000,000

3. Suppose a glacier is melting proportionately to its volume at the rate of 15% per year. Approximately what percent of the glacier is left after ten years if the initial volume is one million cubic meters? (This is an example of exponential decay.)

Solution: Modeling equation: $P = 1000000e^{-0.15 \cdot 10}$ (where P is the volume)
After 10 years: 22.3%

4. A snowball is rolling down a snow covered hill. Suppose that at any time while it is rolling down the hill, its weight is increasing proportionately to its weight at a rate of 10% per second. What is its weight after 10 seconds if its weight initially was 2 pounds? After 20 seconds? After 45 seconds? After 1 minute? What limitations might exist on this problem?

Solution: Modeling equation for growing snowball: $P = 2e^{0.1t}$ (where P is the weight and t is the time in seconds);
After 10 seconds: 5.43 pounds;
After 20 seconds: 14.78 pounds;
After 45 seconds: 180.03 pounds;
After 1 minute: 806.86 pounds

Possible limitations: the modeling equation might not be appropriate after too long a period of time, for example—the snowball may break apart if it gets too big, or it might reach the end of the hill.

Teacher Tip: Students encounter the exponential constant e at various levels in their mathematics schooling. It may happen well before they reach calculus, and it is often used without an appreciation for where it originates (or why it is important). A good time to use this activity is when students first encounter e, but it is also appropriate for Precalculus and Calculus students when they are studying derivatives and instantaneous rate of change.