CALCULUS

with the
TI-89

Sample Activity: Exploration 6

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Isaac Newton 1642 – 1727

Gottfried Leibniz 1646 – 1716
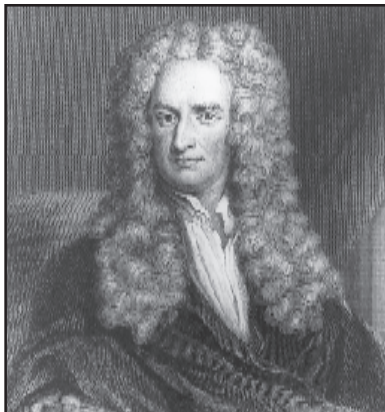
Isaac Newton, Karl Gauss, and Archimedes are rated by historians of mathematics as the three greatest mathematicians of all time. Newton also enjoys such status as a scientist.

In 1665-1666, an outbreak of the bubonic plague forced the closing of Cambridge University. Newton, who had been pursuing his studies there, was sent home to the family farm at Woolsthorpe. There, in an 18-month period of intensely creative work, he made several landmark discoveries that changed the course of mathematics and science. The most mathematically significant of these was his discovery of calculus. To analyse continuously changing quantities which he called *fluents*, (functions of time) he expressed the rate of change of these fluents in terms of other "derived" fluents which he called *fluxions* (and which we now call *derivatives*). Fluents and the techniques for calculating their fluxions (and inverse fluxions) became the basis for a new branch of mathematics which we now call *calculus*. Using calculus, scientists were suddenly able to model in mathematical terms, continuously changing quantities in nature. However, Isaac Newton did not publish his findings in 1666, because an earlier publication on the nature of light had generated such criticism, that Newton vowed never to publish again. Later in life, when a controversy erupted over precedence in the invention of calculus, Newton came to regret his reluctance to publish. Although his discovery of calculus was known to a small cadré of Newton's inner circle in the late 1660's, there was no widely published version of his work until 1704.

Gottfried Leibniz, described by historians of mathematics as a "universal genius", was the son of a professor of moral philosophy. A child prodigy, Gottfried devoured all manner of books on matters mathematical and philosophical, graduating with a doctorate at age 20. His early work in the late 1660's was the quest for a universal language (*characteristica universalis*) and an algebra of reasoning (*calculus ratiocinator*) which would reduce moral questions to a sequence of formal computations. This work anticipated by a century the subsequent development of symbolic logic.

While in Paris in 1672, Leibniz befriended the great Dutch scientist, Christiaan Huygens, who introduced him to the current frontiers of mathematics. Initial interest in summing infinite series, led Leibniz to investigations involving tangents to curves and areas bounded by them. By 1676, Gottfried Leibniz had discovered the fundamental principles of calculus which Newton had discovered a decade earlier, but not yet published. In 1684, Leibniz published "his" calculus in the journal *Acta Eruditorum* of which he was the editor. This triggered a maelstrom of controversy — the English claiming that he plagiarized Newton's calculus and the Continental Europeans claiming independent discovery. Now with the perspective of hindsight, the historians of mathematics generally attribute credit to both men as independent co-inventors of calculus.

Newton's first important mathematical discovery came during the 1665-1666 hiatus from his formal studies. During those precious months at Woolsthorpe, Newton discovered the Binomial Theorem for fractional exponents. In modern notation, the binomial theorem for fractional exponents is the expansion of $(1 + x)^{m/n}$ as an infinite power series.

The Binomial Theorem for Fractional Exponents

$$(1+x)^{\frac{m}{n}} = 1 + \frac{m}{n}x + \frac{\left(\frac{m}{n}\right)\left(\frac{m}{n}-1\right)}{2!}x^2 + \frac{\left(\frac{m}{n}\right)\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{3!}x^3 + \dots$$

The Binomial Theorem enabled Newton to compute $f(x + \Delta x) - f(x)$ for algebraic functions of x and thus calculate their derivatives.

WORKED EXAMPLE 1

a) The distance y meters traveled by a rocket in the first x seconds after launch ($x < 120$) is given by
$$y = 0.08x^3 - 1.5x^2 + 24x$$
Graph the distance vs. time equation and find the distance traveled in the first minute.

b) Write an expression for the distance traveled in the first a seconds.
Write an expression for the average velocity over the time interval $a \leq x \leq a + \Delta x$ where $\Delta x < 1$ and a is any constant such that $0 \leq a \leq 120$.

c) Evaluate the limit of the average velocity over the interval $a \leq x \leq a + \Delta x$ as $\Delta x \to 0$, to obtain the instantaneous velocity when $x = a$ seconds.

SOLUTION

a) To graph this equation, we define:
$$y1(x) = 0.08x^3 - 1.5x^2 + 24x$$
and we set the window variables as shown in the display.

```
xmin=-10.
xmax=120.
xscl=1.
ymin=-10.
ymax=120000.
yscl=1.
xres=2.
```

To find the distance traveled in the first minute, i.e. 60 seconds, we can enter $y1(60)$ on the command line, or select **value** from the F5 menu of the graph screen and enter 60 as in *Worked Example* 1 on page 27. We obtain $y1(60) = 13\ 320$ using either method.

b) The distance traveled in the first a seconds is:
$$y1(a) = 0.08a^3 - 1.5a^2 + 24a$$
The average velocity over the interval $a \leq x \leq a + \Delta x$ is represented on the graph by $\Delta y/\Delta x$; that is, the slope of the secant. PQ

$y1(x) = 0.08x^3 - 1.5x^2 + 24x$
$(a + \Delta x, y1(a + \Delta x)) \to$ Q
P $\to (a,\ y1(a))$
Δy
Δx
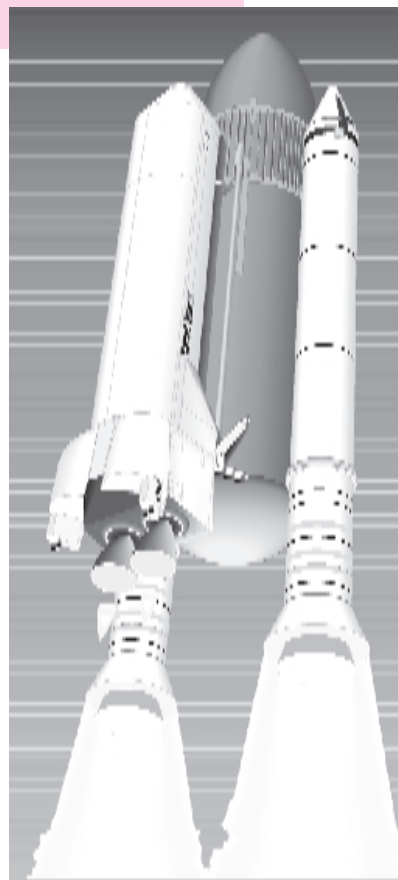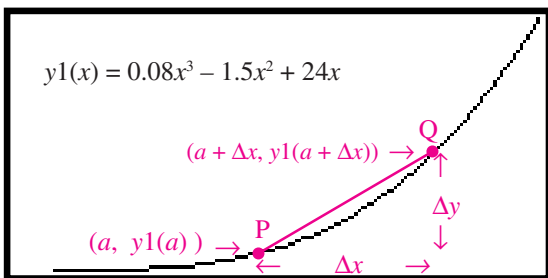
$$\frac{\Delta y}{\Delta x} = \frac{y1(a+\Delta x) - y1(a)}{\Delta x}$$

$$= \frac{[0.08(a+\Delta x)^3 - 1.5(a+\Delta x)^2 + 24(a+\Delta x)] - [0.08a^3 - 1.5a^2 + 24(a)]}{\Delta x}$$

$$= \frac{0.08[3a^2\Delta x + 3a(\Delta x)^2 + (\Delta x)^3] - 1.5[2a\Delta x + (\Delta x)^2] + 24\Delta x}{\Delta x}$$

$$= 0.08(3a^2) - 1.5(2a) + 24 + \Delta x[0.08(3a) - 1.5 + 0.08\Delta x]$$

c) As $\Delta x \to 0$, the point Q slides along the curve toward P and the slope of the secant PQ becomes the slope of the tangent at P. On evaluating the limit as $\Delta x \to 0$, we find
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0.08(3a^2) - 1.5(2a) + 24$$
or $0.24a^2 - 3a + 24$ as shown on the display.

Note: To get Δx on your TI-89 press:

2nd [CHAR] ▶ 5

```
■ lim    ( y1(a + ▵x) - y1(a) )
  ▵x→0   (        ▵x           )
        .24·(a² - 12.5·a + 100.)
...mit((y1(a+▵x)-y1(a))/▵x,...

■ expand( lim    ( y1(a + ▵x) - ↳
          ▵x→0   (      ▵x
        .24·a² - 3.·a + 24.
...1(a+▵x)-y1(a))/▵x,▵x,0))
```

The method used in *Worked Example* 1 is the basis of differential calculus. That is, to find the instantaneous velocity at a point $x = a$, we take the limit of the average velocities over shorter and shorter intervals $a \le x \le a + \Delta x$. Since the instantaneous velocity at $x = a$ is the slope of the tangent to the distance-time graph at $x = a$, this is tantamount to finding the slope of the tangent by taking the limit of the slopes of the secants joining $(a, f(a))$ to $(a + \Delta x, f(a + \Delta x))$ as $\Delta x \to 0$. The expression $0.24a^2 - 3a + 24$, is called the *derivative of* $y1(x)$ *at* $x = a$.

The calculations in *Worked Example* 1 were rather tedious. To simplify the process, we formalize the definition of the derivative.

Definition: The derivative of a function $f(x)$ at $x = a$, denoted by $f'(a)$ is:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

The foregoing definition assumes that the limit exists. If the limit does not exist, the function is said to have no derivative at $x = a$.

WORKED EXAMPLE 2

a) Find expressions for the derivative of each function $f(x)$ at $x = a$.

    i) $f(x) = kx$ ii) $f(x) = kx^2$ iii) $f(x) = kx^3$

where k is any constant.

b) Use your results in part a) to calculate the derivative of the function $y1(x)$ at $x = a$ where $y1(x)$ is the function defined in *Worked Example* 1.

SOLUTION

a) For each function, we construct the quotient $\dfrac{f(a + \Delta x) - f(a)}{\Delta x}$ and then take the limit as $\Delta x \to 0$.

i) $\lim\limits_{\Delta x \to 0} \dfrac{f(a + \Delta x) - f(a)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{k(a + \Delta x) - k(a)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{k\Delta x}{\Delta x}$

$= k$

ii) $\lim\limits_{\Delta x \to 0} \dfrac{k(a + \Delta x)^2 - k(a^2)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{k\left[a^2 + 2a\Delta x + (\Delta x)^2\right] - k(a^2)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{k\left[2a\Delta x + (\Delta x)^2\right]}{\Delta x}$

$= 2ak$

iii) $\lim\limits_{\Delta x \to 0} \dfrac{k(a + \Delta x)^3 - k(a^3)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{k\left[a^3 + 3a^2\Delta x + 3a(\Delta x)^2 + (\Delta x)^3\right] - k(a^3)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{k\left[3a^2\Delta x + 3a(\Delta x)^2 + (\Delta x)^3\right]}{\Delta x}$

$= 3a^2k$

b) In exercise ❷, you will prove that the derivative of a sum of two or more functions is the sum of the derivatives. That is, if $f_1(x)$ and $f_2(x)$ are two functions, then $(f_1 + f_2)'(a) = f_1'(a) + f_2'(a)$. Therefore setting $f_1(x) = 0.08x^3, f_2(x) = -1.5x^2$ and $f_3(x) = 24x$;

    $y1'(a) = 3(0.08)a^2 - 2(1.5)a + 24$,

or $y1'(a) = 0.24a^2 - 3a + 24$, as before.

To obtain the derivative on your TI-89, press: **F3** **ENTER**

Then enter the commands as shown on the command line of the display. You obtain the derivative as shown.

NOTE:
THE PROCESS OF FINDING THE DERIVATIVE OF A FUNCTION IS CALLED DIFFERENTIATION.

$\blacksquare \dfrac{d}{da}\left(.08 \cdot a^3 - 1.5 \cdot a^2 + 24 \cdot a\right)$

$.24 \cdot a^2 - 3. \cdot a + 24$

d(0.08a^3-1.5a^2+24a,a)

❶. a) State in your own words what is meant by the *derivative of a function f (x) at x = a.*

b) What does the derivative of $f(x)$ at $x = a$ tell you about the graph of $f(x)$ at $x = a$?

❷. Suppose $f_1(x)$ and $f_2(x)$ are two functions which have a derivative at $x = a$. Use the definition of the derivative to prove that the derivative of $f_1(x) \pm f_2(x)$ at $x = a$ is
$$f_1'(x) \pm f_2'(x).$$
(Assume that the limit of a sum is the sum of the limits.)

❸. Find the derivative of each function at $x = 3$.
a) $f(x) = 4x^2 - 7$ b) $f(x) = 5x^3 + 6x$
c) $f(x) = 2x^3 + 3x^2 - 7$ d) $f(x) = 8x^3 + 4x - 9$

❹. For each of the functions in exercise ❸, find the equation of the tangent to the curve $y = f(x)$ at $x = 3$.

❺. In *Worked Example* 2 p. 28, we saw that the distance y traveled by the VIPER sportscar t seconds into the race is given by the equation:
$$y = -0.07t^3 + 3.15t^2 + 1.2t$$
Calculate the velocity of the VIPER at $t = 5.5$ seconds. Compare with your answer to exercise ❾e) on page 29.

❻. The TI-89 has command **Tangent** on the F5 menu of the graph display. To find the tangent to the graph of $f(x) = 2x^3 + 7x^2 - 4$, at $x = 4$, we graph $f(x)$ in the window:
$$-10 \le x \le 10; \quad -1000 \le y \le 1000$$
Then we press: **F5** **alpha** **A** to obtain a display like this:


```
                                1
          Tangent at?
          xc:.126582    yc: -3.88378
```
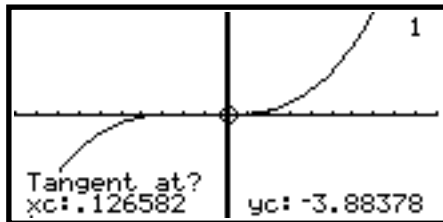
In response to the prompt **Tangent at?**, we enter 4. After pressing **ENTER** we obtain this display.


```
          y=152.x-372.
```
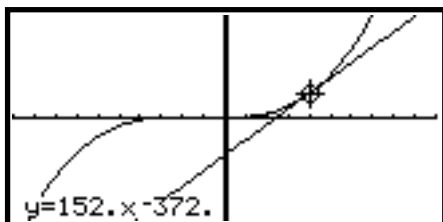
The display gives the equation of the tangent at $x = 4$; that is,
$$y = 152x - 372$$
Use the **Tangent** command on your TI-89 to verify your answers in exercise ❹.

❼. In our definition of the derivative of $f(x)$ at $x = a$, we took the limit over the interval $a \le x \le a + \Delta x$ as $\Delta x \to 0$. Alternatively, we could have defined the derivative of $f(x)$ at $x = a$ as the limit over the interval $a - \Delta x \le x \le a + \Delta x$ as $\Delta x \to 0$. This is called the *central difference quotient.*
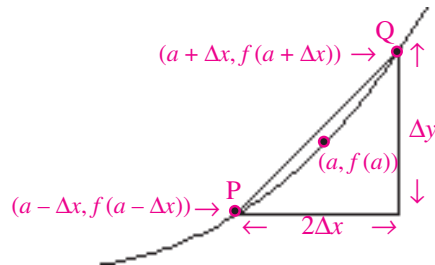
In formal terms, the difference quotient definition of the derivative of $f(x)$ at $x = a$ is:
$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a - \Delta x)}{2\Delta x}$$

a) Explain why you might expect the difference quotient definition of $f'(a)$ to be equivalent to the definition of $f'(a)$ given on page 32.

b) The calculus menu (**F3**) of your TI-89 has the command **nDeriv(** (short for "numerical derivative"). For any function $f(x)$, **nDeriv(f (x), x, h)** yields:
$$\frac{f(a+h) - f(a-h)}{2h} \quad \leftarrow\text{Note that } \Delta x \text{ is replaced by } h.$$
This command automatically forms the different quotient for us. To access **nDeriv(** for $f(x)$, press:
F3 **alpha** **A** . Then enter $f(x)$, x, and h as in the display.


```
■ nDeriv(f(x),x,h)
              -(f(x - h) - f(x + h))
              ──────────────────────
                       2·h
nDeriv(f(x),x,h)
```
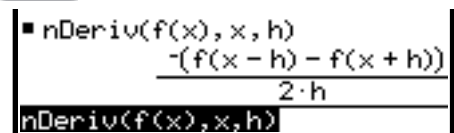
In the command line of the home screen, replace $f(x)$ by x^m $(m > 0)$. Then take the limit as $h \to 0$, as shown in the display.

```
limit(nDeriv(x^m,x,h),h,0)
```

c) Write your answer in simplest form. Use your answer in part b) to conjecture a formula for $f'(a)$ when $f(x) = x^m$. Apply your formula to the functions in *Worked Example* 2 on page 32. Compare with the answers given there.

d) To obtain the derivative of $f(x) = x^m$ at any point x, press:
F3 **ENTER** and enter the commands as shown below.

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d(x^m,x)
```

e) Use the binomial theorem given on page 30 to expand
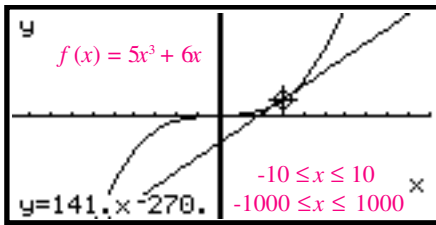$$\frac{(x + \Delta x)^m - (x - \Delta x)^m}{2\Delta x} \quad \text{where } m \text{ is a positive integer}$$
Then take the limit as $\Delta x \to 0$ to find $f'(x)$ for $f(x) = x^m$.
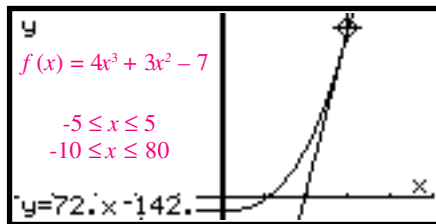
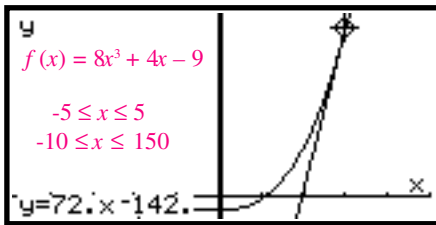3a) & 4a) The graph of $y = 4x^2 - 7$ and its tangent line at $x = 3$.
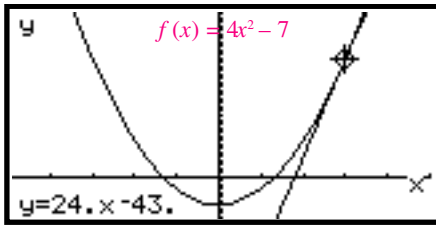
3b) & 4b) The graph of $y = 5x^3 + 6x$ and its tangent line at $x = 3$.

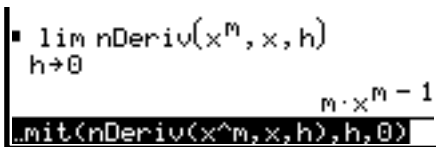3c) & 4c) The graph of $y = 4x^3 + 3x^2 - 7$ and its tangent line at $x = 3$.

3d) & 4d) The graph of $y = 8x^3 + 4x - 9$ and its tangent line at $x = 3$.

5. The velocity is given by $dy/dx = -0.21t^2 + 6.3t + 1.2$, so the velocity at $t = 5.5$ is $29.4975 m/s$.

6. See the displays above in the answers to exercises 3 and 4.

7. b) We obtain the following display.

c) Substitution of a for x yields $f'(a) = ma^{n-1}$.

d) We obtain the display below.