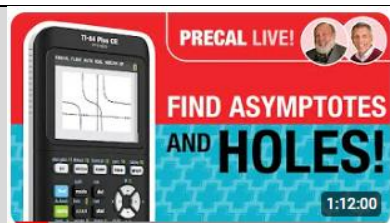**Thursday Night Precalculus Series**
September 28, 2023
In this AP Precalculus Live session, we will explore several examples on finding zeros, holes, vertical asymptotes, and end behavior of rational function.

Rational Functions: Zeros, Holes, Vertical Asymptotes, and End Behavior

About the Lesson
- This Teacher Notes guide is designed to be used in conjunction with the AP Precalculus Live session and Student Problems document that can be found on-demand:
  https://www.youtube.com/watch?v=8gvlFvSazDI&list=PLQa_6aWmaC6B-5h5n2Cr5h3G2ZPfJ0HGI&index=1&t=22s
    - *Please note that not all problems/content from the Student Problem Sheet is covered in the video component. Student/Teacher Notes are also useful without students viewing the "Live Session" but can be enriched by that resource.*
- This session involves exploring features of the graphs of rational functions, such as:
    - Using limits to describe end behaviors.
    - Determining the zeros.
    - Determining horizontal, slant, and vertical asymptotes.
    - Determining holes in the graphs of rational functions.
- Students should be able to use the TI-Nspire to verify these features of a rational function.

- ✦ **Class Discussion**: Use these questions to help students communicate their understanding of the problem. These questions are presented in the *Live* video as well.

Materials:

TI-Nspire document
- Rational Functions.tns
Student document
- problems_09_28_23
Solutions
- problems_solutions_09_28_23
YouTube
- https://www.youtube.com/watch?v=8gvlFvSazDI&list=PLQa_6aWmaC6B-5h5n2Cr5h3G2ZPfJ0HGI&index=1&t=22s
- ***Documents and materials can be downloaded from this site.***

AP Precalculus Learning Objectives
- 1.7.A: Describe end behaviors of rational functions.
- 1.8.A: Determine the zeros of rational functions.
- 1.9.A: Determine asymptotes of graphs of rational functions.
- 1.10.A: Determine holes in graphs of rational functions.

Source: AP Precalculus Course and Exam Description, The College Board

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Problem 1. (a)

Find the location of any zeros and holes for the graph of the given rational function.

$$f(x) = \frac{x^3 - x^2 - 10x - 8}{x + 2}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.

Teacher Note: Some Algebra 2 (or equivalent) courses cover synthetic division. Synthetic division is not in the AP Precalculus curriculum and will not be assessed. Factoring polynomials without technology is limited to factoring out a common factor and the rules for quadratics.

📌 **Class Discussion:**

Why does the graph look like a quadratic?

Possible Answers:

When the common factor is divided out, the remaining function in the numerator is a quadratic.

Or, the ratio of the leading terms, $\dfrac{x^3}{x} = x^2$ is a quadratic function.

📌 **Class Discussion:**

How could you find the $y-$value of the hole?

Possible Answers:

To find the $y-$value of the hole, substitute $x = -2$ into the reduced rational function

$$f(x) = \frac{(x-4)(x+1)}{1}, \ x \neq -2. \ \text{Hole: } (-2, 6)$$

Teacher Note: Revisit the two Class Discussion questions above when using the TI-Nspire.

Problem 1. (b)

Find the location of any zeros and holes for the graph of the given rational function.

$$f(x) = \frac{(x+3)^2(x+1)(x-1)(x-4)}{x^2 + 2x - 3}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.

📌 Class Discussion:

For $f(x) = \dfrac{(x+3)^2(x+1)(x-4)}{(x+3)(x+1)}$, would the graph be different if the factor of $(x+3)$ were

squared in the denominator instead of the numerator? $\left[g(x) = \dfrac{(x+3)(x+1)(x-4)}{(x+3)^2(x+1)} \right]$

Possible Answers:

The graph would now have a vertical asymptote instead of a hole at $x = -3$.

Teacher Note:

Explore the function $g(x)$ using technology.

📌 Class Discussion:

How could you find the $y-$values of the two holes?

Possible Answers:

The hole at $x = -3$ appears to be the location where the graph is attempting to cross the x-axis. To find the $y-$value of the hole at $x = 1$, substitute $x = 1$ into the reduced rational function

$f(x) = \dfrac{(x+3)(x+1)(x-4)}{1}, \ x \neq -3, 1.$ Holes: $(-3, 0)$ $(1, -24)$

Teacher Note: Revisit the Class Discussion question above when using the TI-Nspire.

Using the TI-Nspire with Problem 1. (a)

In the video, all technology demonstrations use the TI-84. The "Using the TI-Nspire" notes in this document provide directions and screen shots that are similar to the use of the TI-84.
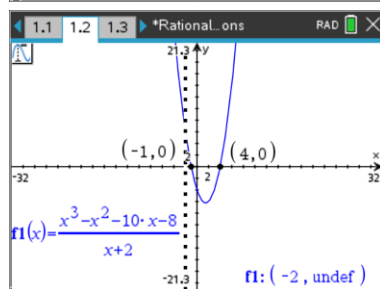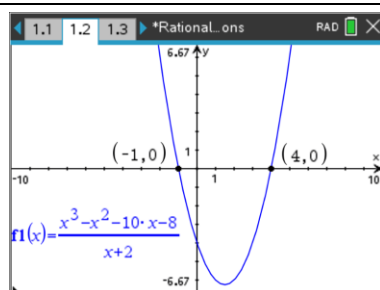

Find the location of any zeros and holes for the graph of the given rational function.

$f(x) = \dfrac{x^3 - x^2 - 10x - 8}{x+2}$

Use the graphing application to confirm the zeros and the hole in the graph of the function.

Try Zoom Decimal for the graph. Find the zeros using Menu – Analyze Graph – Zero.

To confirm the hole at $x = -2,$ set the Trace Step to 1 and then use Trace to move to $x = -2.$

Teacher Note: Zoom Decimal is frequently used to show holes in the graph on the TI-84. Holes do not appear on the TI-Nspire even with the Zoom Decimal window.

Technology Tip: We could also look at a table of values of the function to confirm the locations of the hole and the intercepts. [ctrl] [T] will create a split screen with the table on the right. Scroll up or down to confirm the zeros and the location of the hole. Notice the Undefined with $x = -2.$
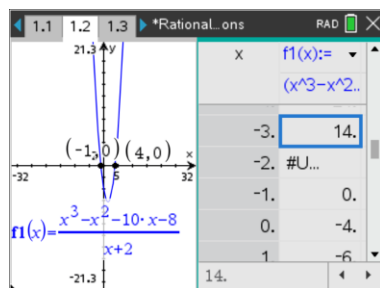
📌 **Class Discussion:**

How could you use the table to get close to $x = -2$ to approximate the $y-$value of the hole?

What is the correct limit notation to show that the function has a hole at $x = -2?$
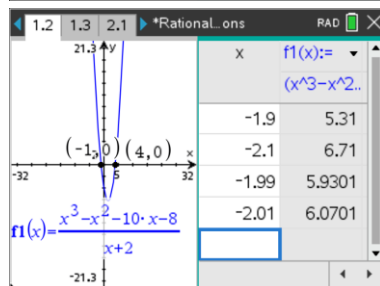
Possible Answers:

Select [menu] - Table – Edit Table settings. Change Independent to Ask. Use $x-$values close to $x = -2.$ Some suggested values are $x = -1.9$ and $x = -2.1.$ Also try $x = -1.99$ and $x = -2.01.$ Students should observe that the $y-$values are getting close to 6.

$$\lim_{x \to -2} f(x) = 6$$

Technology Tip: Press [ctrl] [T] again to go back to the graph only.

Using the TI-Nspire with Problem 1. (b)

The use of the calculator for this problem is not discussed in the video.

Find the location of any zeros and holes for the graph of the given rational function.

$$f(x) = \frac{(x+3)^2(x+1)(x-1)(x-4)}{x^2+2x-3}$$

Use the graphing application to confirm the zeros and the hole in the graph of the function.

Try Zoom Decimal for the graph. Find the zeros using Menu – Analyze Graph – Zero.

To confirm the hole at $x = -3,$ set the Trace Step to 1 and then use Trace to move to $x = -3.$

📌 **Class Discussion:**

Does it make sense that the rational function looks like the graph of a cubic polynomial?

Possible Answers:

When the common factor is divided out, the remaining function in the numerator is cubic. Or, the rational function
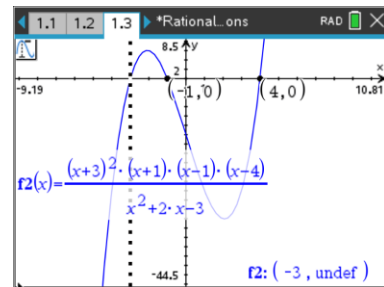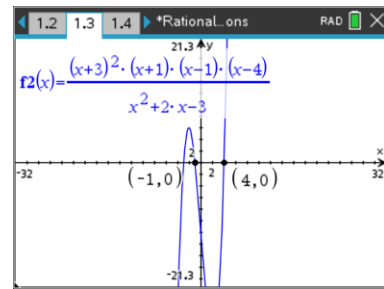
$$f(x) = \frac{(x+3)^2(x+1)(x-1)(x-4)}{x^2+2x-3}$$ has a polynomial of degree

5 in the numerator and degree 2 in the denominator. The ratio of

leading terms $\dfrac{x^5}{x^2} = x^3$ is a cubic polynomial.

Teacher Tip: Discuss the appearance of an $x-$ intercept at $x = -3,$ and why there is a hole in the graph of the function. Take a look at a table of values.

Review how to find the $y-$ value of the hole when $x = 1.$

Problem 2. (a)

Find all the vertical asymptotes for the graph of the given rational function and sketch a complete graph.

$$f(x) = \frac{x^3 - 5x^2 + 6x}{x^2 - 9}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem

📌 **Class Discussion:**

What is the behavior of the function as the graph approaches the vertical asymptote, $x = -3$, on either side?

Possible Answers:

As the graph approaches –3 from the left, the y-values approach negative infinity.

$$\lim_{x \to -3^-} f(x) = -\infty$$

As the graph approaches –3 from the right, the y-values approach positive infinity.

$$\lim_{x \to -3^+} f(x) = \infty$$

📌 **Class Discussion:**

What is the y-coordinate of the hole?

Possible Answers:

To find the $y-$value of the hole at $x = 3$, substitute $x = 3$ into the reduced rational function

$$f(x) = \frac{x(x-2)}{x+3}, \; x \neq -3. \quad \text{Hole: } \left(-3, \frac{1}{2} \right)$$

Problem 2. (b)

Find all the vertical asymptotes for the graph of the given rational function and sketch a complete graph.

$$f(x) = \frac{x^2 - 4}{(x-2)(x^2 - 6x + 5)}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.

📌 **Class Discussion:**

Can the graph of a rational function cross its vertical asymptote?

Possible Answers:

No, it cannot.

Using the TI-Nspire with Problem 2. (a)

Use the graphing application to confirm the behavior of the graph of the function near the vertical asymptote.

The top graph is using Zoom Decimal. Use Trace to explore the zeros and the hole at $x = 3$.

📌 **Class Discussion:**

Does the graph "dip" below the $x-$axis?

What are the coordinates of the hole?

Possible Answers:

Yes, the graph dips below the $x-$axis. Students should Zoom in on the graph to clearly see that the graph goes below the $x-$axis between $x = 0$ and $x = 2$.

Use the graph or table to explore values of the function near $x = 3$.

The $y-$coordinate of the hole is $\dfrac{1}{2}$. Using a limit, we would write

$$\lim_{x \to 3} f(x) = \frac{1}{2}.$$

To explore the behavior near the vertical asymptote, adjust your window so that the graph fits on the screen. A y-min of -20 and a y-max of 20 works well.
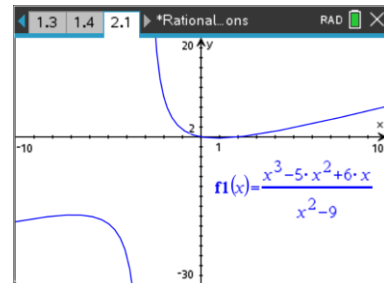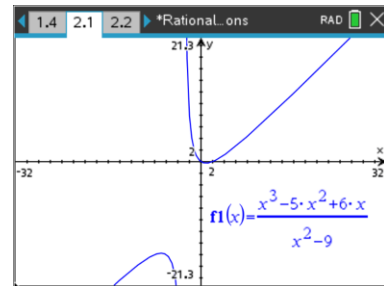
📌 **Class Discussion:**

How do we use limit notation to describe the behavior of the function as the graph approaches -3 from both the left and the right?

Possible Answers:

$$\lim_{x \to -3^-} f(x) = -\infty$$
$$\lim_{x \to -3^+} f(x) = \infty$$

Using the TI-Nspire with Problem 2. (b)
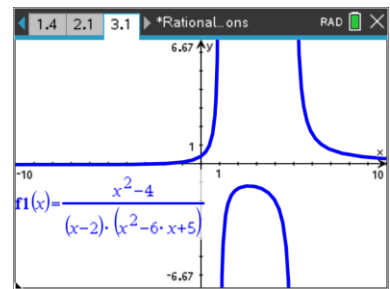*The use of the calculator for this problem is not discussed in the video.*

Use the graphing application to confirm the behavior of the graph of the function near the vertical asymptotes.

Class Discussion:

Use limit notation to describe the behavior of the graph on either side of the vertical asymptotes.

Possible Answers:

$$\lim_{x \to 1^-} f(x) = \infty \qquad\qquad \lim_{x \to 5^-} f(x) = -\infty$$

$$\lim_{x \to 1^+} f(x) = -\infty \qquad\qquad \lim_{x \to 5^+} f(x) = \infty$$

Using Sliders on the TI-Nspire to demonstrate Transformations.
Here is a Youtube Video in the Texas Instruments Education Channel on creating sliders.

https://youtu.be/FM5wpN7t-lk?si=8JyFjGK3JmpHsRb3

Technology Tip: Use sliders to investigate rational functions.

On a graphing application, type $f(x) = \dfrac{(x-a)(x-b)}{(x-c)(x-d)}$. Select

OK. Moving and/or minimizing the sliders makes them easier to use.

a, b, c, and d are constants that will be used as parameters.

Class Discussion:

When $a = 1$, $b = 1$, $c = 1$, and $d = 1$, why does the graph appear to be the horizontal line $y = 1$?
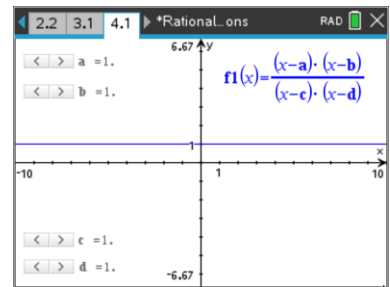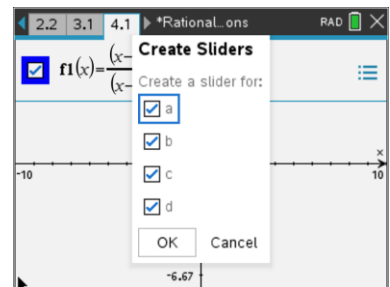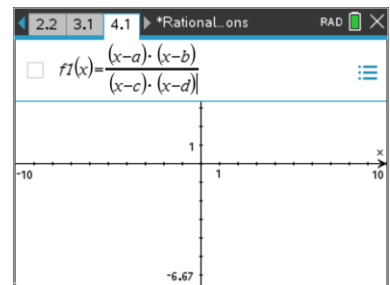
Possible Answers:

The function is not defined at $x = 1$. The numerator and denominator are exactly the same, so the function is equivalent to 1 with $x \ne 1$.

Class Discussion:

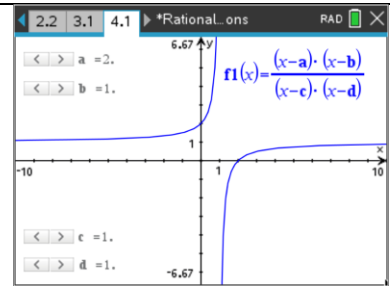If $a = 2$, $b = 1$, $c = 1$, and $d = 1$, what do you notice about the graph of the function?

Possible Answers:

The function has a zero at $x = 2.$ We have three linear factors of $x - 1,$ one in the numerator and two in the denominator. Because we have more factors of $x - 1$ in the denominator than the numerator, we have a vertical asymptote $x = 1.$

📌 **Class Discussion:**

How could we change the values of $a,$ $b,$ $c,$ and d to have a hole at $x = 1?$

Possible Answers:

One possibility is $a = 3,$ $b = 1,$ $c = 2,$ and $d = 1.$

Teacher Tip: Sliders allow the teacher to "jeopardize" questions in class. Challenge the students to create a graph with certain characteristics.

📌 **Class Discussion:**

How could we change the values of $a,$ $b,$ $c,$ and d to have a zero at $x = 1,$ a hole at $x = 2,$ and a vertical asymptote at $x = 3?$

Possible Answers:

One possibility is $a = 1,$ $b = 2,$ $c = 2,$ and $d = 3.$ $f(x) = \dfrac{(x-1)(x-2)}{(x-2)(x-3)}.$

Technology Tip:

Graph horizontal asymptotes as functions using the function entry. Graph vertical asymptotes using the graph relations feature.

Note: *The following problems are not discussed in the video.*

Problem 3. (a)
Express the end behavior of each rational function using limit notation and sketch a complete graph.
$$f(x) = \frac{10x}{x^2 + 4}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.

Problem 3. (b)

Express the end behavior of each rational function using limit notation and sketch a complete graph.

$$f(x) = \frac{3x^2 - 15}{x^2 - 2x - 8}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.

Problem 3. (c)

Express the end behavior of each rational function using limit notation and sketch a complete graph.

$$f(x) = \frac{x^3 - x}{x^2 - 4}$$

Sample Solution:

Refer to the Teacher Solutions Document for the full solution to this problem.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

• The graphing application can be used to explore zeros and holes.

• The graphing application can be used to explore the behavior of a function near the vertical, horizonal, and slant asymptotes.

• The use of tables on the calculator also supports conclusions about the behavior of rational functions.

For more videos from the AP Precalculus Live series, visit our playlist
https://www.youtube.com/playlist?list=PLQa_6aWmaC6B-5h5n2Cr5h3G2ZPfJ0HGI

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