Chapter 4

Patterns, Functions, and Algebraic Reasoning

In this chapter

*This chapter summarizes NCTM's **Principles and Standards for School Mathematics** for patterns, functions, and algebraic reasoning and applies those principles in some activity overviews. You will learn how to:*

- *Use the calculator to generate recursive patterns.*
- *Help students reflect on how they learn mathematics.*

Overview of patterns, functions, and algebraic reasoning

In the document *Principles and Standards for School Mathematics* (NCTM, 2000) the topics of patterns, functions, and algebraic reasoning are primarily described under the strand of Algebra. As with many topics in mathematics, it is difficult to separate the content strand from the process stands. Hence, many examples of patterns and functions can also be identified in the strands of number and operations, geometry, data analysis and probability, as well as problem solving, reasoning and proof, communication, connections, and representation.

Goals for students

Instructional programs from prekindergarten through grade 12 should enable all students to:

- understand patterns, relations, and functions;
- represent and analyze mathematical situations and structures using algebraic symbols;
- use mathematical models to represent and understand quantitative relationships;
- analyze changes in various contexts. (NCTM, 2000, page 37)

Early experiences that provide children with the opportunity to employ the processes of classifying and ordering can naturally lead to expressing patterns either with pictures or orally. It is common for children to express the relationship in each of the following sequences: 2, 4, 6, 8, … and 1, 3, 5, 7, … as *add two*. In doing so, they are focusing on how each term is obtained from the previous term, an important component of recursive thinking. Later, however, students need to be able to express these two sequences as *2n*, and *2n – 1*, respectively. As students explore patterns and relations in the middle grades, they should be able to generate and validate equivalent expressions.

Goals for teachers

Along with considering the mathematics elementary school children are to learn, you must also consider what mathematical content teachers of elementary school mathematics learn. While examining the appropriate content for teachers, the Conference Board of Mathematical Sciences (CBMS) notes, "Those who prepare prospective teachers need to recognize how intellectually rich elementary-level mathematics is. At the same time, they cannot assume that these aspiring teachers have even been exposed to evidence that this is so." (CBMS, 17) It becomes important for instructors of prospective teachers to engage their students in studying these complexities and having them reflect on how they learn mathematics. In discussing the role of algebra and functions for prospective teachers of mathematics at the elementary level, CBMS states:

> Algebraic notation is an efficient means for representing properties of operations and relationships among them. In the elementary grades, well before they encounter that notation, children who are encouraged to recognize and articulate generalizations will become familiar with the source of ideas they are later to express algebraically. In order to support children's learning in this realm, teachers first must do this work for themselves. Thus, they must come to recognize the importance of generalization as a mathematical activity. In the context of number theory explorations (for example, odd and even numbers, square numbers, factors) they can look for patterns, offer conjectures, and develop arguments for the generalizations they identify. Moreover, the arguments they propose become the occasions for investigating different forms of justification. If, in this work, teachers learn to use a variety of modes of representation, including conventional algebraic symbols, the algebra they once experienced as the manipulation of opaque symbols can be invested with meaning. (CBMS, 20)

Teachers of mathematics at the middle level must have a deeper understanding of concepts and procedures related to algebra and functions. The scope of algebra at the middle level expands so teachers must be able to move comfortably between and among differing general ideas related to algebra. The Conference Board for Mathematical Sciences suggests "Prospective middle grades teachers should understand and be able to work with algebra as a symbolic language, as a problem solving tool, as generalized arithmetic, as generalized quantitative reasoning, as a study of functions, relations, and variation, and as a way of modeling physical situations." (CBMS, 20)

The use of calculators in teaching and learning patterns, functions, and algebraic reasoning

A calculator can be successfully used to address several of the components of the content mentioned above. With appropriate use of a constant key, a calculator can be used to generate recursive patterns. By using patterns as simple as adding one constantly, children can learn counting. When a pattern has been determined, a calculator with graphing capability can be used to give a visual picture of the pattern. Furthermore, a calculator with statistical capability can be used to generate
an equation for the pattern. Of course, a calculator can be used to perform computations related to predictions or to extend patterns that would not be made otherwise.

While the ideas suggested here begin with patterns, they also show the natural connection that can be made from patterns to functions and algebraic reasoning. The explorations model how to examine the idea of a variable for an unknown quantity and to express mathematical relationships using equations. They also use representations such as graphs and tables to draw conclusions. The activities also provide an opportunity to examine how changes in a variable may influence the solution to a problem. Several of these explorations provide an opportunity to compare different forms or representations for a function with an emphasis on linear relationships.

Sample Activities

The activities below are intended to provide a mechanism for prospective teachers to learn mathematics, reflect on what they have learned, and reflect on how to teach these same ideas to elementary school students. Children recognize, generate, and explore patterns before they enter school. Many times these patterns can be turned into number patterns that provide an avenue for further arithmetic explorations. As you explore the problem below, consider how a calculator may be used to find a solution.

Activity 1: Guess My Number

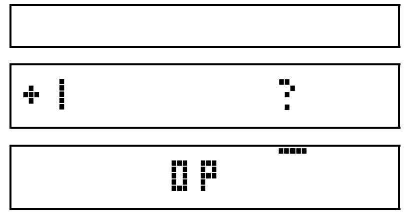
This example is provided to illustrate that children at grade 1 can become familiar with the computational features of a calculator for exploring problems involving algebraic relationships.

The TI-15 can be used to get children familiar with the operation function, to look at reversibility, and to use the calculator as a function machine.

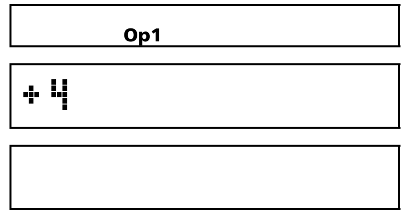
1. Press [On/Off] and [Clear] simultaneously to reset the calculator. Resetting the calculator returns the settings to their defaults and clears pending operations.

Note: On the TI-10*, press* [AC] *to reset the calculator.*

2. Press (Mode). Press ◄ and then ► to underline ?. Press [Enter].



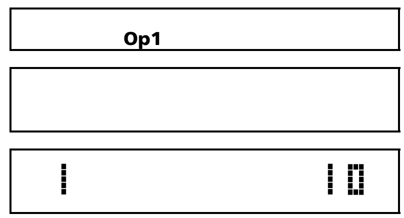
3. Press [Op1] [+] **4** [Op1].
4. Press (Clear).



After making these settings, the following examples can be given to children.

1. Press **6** [Op1]. Observe the outcome: **1 10**.

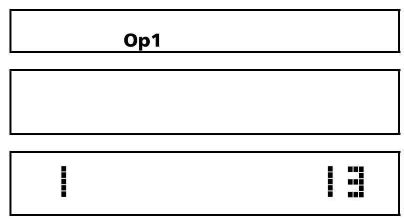
What did **Op1** do?



2. Press **9** [Op1]. Observe the outcome: **1 13**.

What did **Op1** do?

In each example, **Op1** appears to be adding 4 to each entry.



This is a good introduction to the meaning of *operation*. That is, you entered a number like 6 and then **Op1** operated on that number to produce a result of 10. If you have correctly determined that **Op1** was *adding four*, then you can pursue another line of investigation.

For example, if you wanted **Op1** to produce a result of 15, what number would you use at the beginning? (11) This question asks children to reverse their thinking and is similar to the question ____ + 4 = 15.

Activity 2: Connecting Cubes

Becoming familiar with the use of the **Op1** and **Op2** functions on the TI-15 or constant keys on other calculators will be useful in exploring the activities that follow. This activity shows how looking at patterns when exploring with cubes can also be investigated with a calculator.

Angelica was making a train with connecting cubes. She began making her train using the following pattern of colors (**R** stands for red, **B** for blue, **Y** for yellow, and **G** for green):

R R B R R B R R B R R B R

Answer questions 1 - 5 as a *learner* of mathematics.

1. If Angelica continues this pattern, what will be the color of the 20th cube? The 88th cube? The 147th cube?
2. Describe how you solved the problem. Can you think of another way to solve the problem?
3. How are the different methods of solution similar? How are they different?
4. What do you notice about the location of the blue cube in the pattern? How does this help you determine the color for a cube no matter in which place it is? How can you relate this to skip counting by threes or use of multiples of three?
5. How would your solution change if the colors were **B R Y B R Y B R Y**? If they were **R R Y B R R Y B R R Y B**? How will you organize your work for this? Will you use ideas that you heard from others when discussing the previous question? Describe at least three patterns you see in each.

Answer questions 6 - 10 as a *teacher* of mathematics.

6. How could you encourage children to use a variety of approaches to solve this problem? How could you help a child to see that the **B** cubes are related to multiples of 3?
7. How could children use the skip counting ability of a calculator to solve this problem?
8. How would you expect children to know 147 is a multiple of 3?
9. How could you make use of the following pattern to help children see not only multiples of three or four, but skip counting by three or four?

   B R Y B R Y B R

10. How could you make use of the following pattern to help children see not only multiples of three or four, but skip counting by three or four?

    R R Y B R R Y B R R Y B

Answers and Comments

Question **1**

The 20th cube will be red, the 88th cube will be red, and the 147th cube will be blue.

Questions **2 – 4**

While there are different ways to solve the problem, it can be seen that the position of the blue cubes are always in spots that are multiples of 3. That is, the 3rd, 6th, 9th, and so forth, cubes are blue. Therefore, if the position is a multiple of 3, the cube is blue. Otherwise, the cube is red. The differing ways of seeing the patterns need to be compared as they arise.

Question **5**

The B R Y B R Y B R Y pattern is essentially the same as the original, except that three colors are used. The yellow cubes occur at multiples of 3. The R R Y B R R Y B pattern repeats in blocks of four. Here the blue cubes occur at multiples of 4. The first two after a multiple of 4 are red, and the cube right before the multiple of 4 is yellow. The different patterns need to be compared as they arise.

Questions **6 – 8**

Because children will see various ways to determine the colors of the cubes, there are several ways to check on their understanding in this problem. Children should compare and contrast their methods to see if in each case they all arrive at the same conclusions. If appropriate, you may wish to mention that one way of expressing the relationship is that the position of cubes that are multiples of three will be blue, and the other positions will be red. One way to make use of this information is to divide the position number, like 20, by three. If there is no remainder, that number is a multiple of 3 and then that position will be blue. Some students may also know how to use the divisibility rule for three to determine the color.

Question **9**

For the pattern **B R Y B R Y B R . . .** you see that:

- The numbers of the blocks that are blue are: 1, 4, 7, 10, … (That is, these numbers when divided by 3 leave a remainder of 1.)
- The numbers of the blocks that are red are: 2, 5, 8, 11, … (That is, these numbers when divided by 3 leave a remainder of 2.)
- The numbers of the blocks that are yellow are: 3, 6, 9, 12, … (That is, these when divided by 3 leave a remainder of 0. Another way to say this is these numbers are multiples of 3.)

Question **10**

For the pattern **R R Y B R R Y B R R Y B . . .** you see that:

- The numbers of the blocks that are red are: 1, 2, 5, 6, 9, 10, ...
  [or are 1, 5, 9, … or, 2, 6, 10, ...]. (That is, these numbers when divided by 4 leave either a remainder of 1 or 2.)

- The numbers of the blocks that are yellow are: 3, 7, 11, 15, …… (That is, these numbers when divided by 4 leave a remainder of 3.)
- The numbers of the blocks that are blue are: 4, 8, 12, 16… (That is, these numbers when divided by 4 leave a remainder of 0.) Another way to say this is these numbers are multiples of 4.

Students will likely determine different methods for these patterns and should compare results using each pattern. Additionally, for the pattern **B R Y B R Y . . .**, with the use of the integer divide key (Int÷) on the TI-15 or the divide key (÷) on the TI-10, the color of each position can be obtained by checking the remainder when the position number is divided by 3. The position numbers with remainders 0, 1, and 2, will be yellow, blue, and red, respectively.

The colors of the cubes in the pattern **R R Y B R R Y B . . .** can be determined by using Int÷ or ÷, dividing the position number by 4, and checking the remainder. A remainder of 0, 1, 2, or 3 will be blue, red, red, and yellow, respectively.

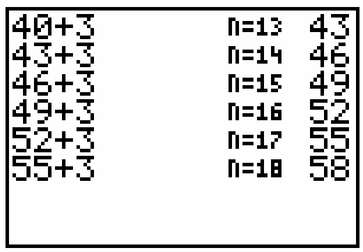
The connection to the use of Int÷ or ÷ for this problem is related to the patterns in the numbers of the cubes that are related to each color. As mentioned earlier, in the pattern **R R Y B R R Y B . . .** cubes numbered 3, 7, 11, 15, 19 **. . .** are the ones that are yellow. Each of these numbers fits the criteria that when you divide it by 4 (using Int÷ or ÷) you will obtain a remainder of 3. Children frequently describe this sequence as starting by 3 and adding 4.

There may be other strategies that can be used to determine the result.

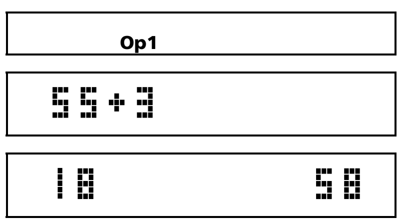
Activity 3: Skip Counting

Children learn how to skip count very early in their school experience. This activity demonstrates the significance of skip counting related to investigations of patterns, reversibility, and algebraic explorations. This activity has direct implication for the use of numbers you expect children as early as grade 1 to examine. At grade 1, children are expected to know such things as, "If you have 13¢ and I give you a nickel, how much money will you have?" "What if I give you three nickels, how much money will you have?" This line of reasoning involves skip counting starting at some fixed point. The sequence 13, 18, 23, 28, … is related to skip counting by 5 starting with 13. Teachers can make a stronger use of skip counting if they do more than study skip counting of multiples. That is, the two sequences 5, 10, 15, 20, … and 13, 18, 23, 28, … have similarities as well as differences that are worthy of discussion.

Calculators handle skip counting in different ways and this exercise assumes knowing how to skip count with a calculator. Skip counting by 3 starting at 4 would produce the screen shown below on the TI-73. Observe the last line: 55+3 n=18 58.



That is, if you start at 4 and skip count by 3 eighteen times, the result is 58. If an operator function (**Op1** or **Op2**) was used on a TI-10 orTI-15, the same information would be provided but the display would reside on two lines, as shown below.



You should practice with the constant key or operation function before exploring this problem.

Roshawn was practicing skip counting by using the constant key (or operator function). Unfortunately while she was pushing buttons she did not always record the appropriate amount of information to recall what she had done. In fact, before she realized it she had written down the result of six different skip counting practice trials. She listed these in **A - F** in the table on the next page. However, as can be seen, for each trial she only wrote down the two numbers showing last on the calculator display. That is, she did not write down other possible useful information such as what numbers were being used to skip count and from what numbers the skip counting began.

Answer questions 1 - 12 as a *learner* of mathematics.

1. For each of the pairs of values below, can you help determine what number Roshawn was using to skip count and also the number where she started?

Part	Number of Times Skip Counting Performed	Result After Skip Counting
A	21	177
B	15	99
C	22	158
D	19	481
E	32	99.2
F	36	5

2. Discuss with others how you attempted question **1**. Is there more than one solution for each problem?

3. Did you think about skip counting by 0 or 1? How would you respond to someone who said counting by 0 or by 1 was skip counting?

4. How could you use working backward to find the starting value?

5. How is this activity similar to the difficulties that first grade children have when confronted with the computation 3 + ___ = 8?

6. Is there a key on the calculator that could be useful for determining a solution?

7. For Part **A**, how is the result of the following keystrokes related to one possible solution?

	Keystroke	Result
TI-10	**177** [÷] **21** [Enter]	**8 r 9**
TI-15	**177** [Int÷] **21** [Enter]	**8 r 9**

Although this produces one possible solution for Roshawn, investigate the problem until other solutions are found.

8. The table on the next page lists all the non-negative integer-valued entries that can be found for the start and skip for Part **A**. What relationships do you notice in the table that refers to the problem in Part **A (21 177)**? What equation expresses the relationship between the numbers in the two columns?

(Hint: the number in the **Skip count by** column multiplied by 21 and added to the number in the **Start at** column always produces 177.) If you could skip count by 6.5, at what value would you start? If you start at 17.3, what value would you use to skip count?

Start at	Skip count by
9	8
30	7
51	6
72	5
93	4
114	3
135	2
156	1
177	0

9. Use the calculator to explore the relationships between the numbers for Parts **B - F**.

10. How do you have to adjust your thinking for Parts **E** and **F**?

11. Can you find one solution to Part **E** by using the following keystrokes?

TI-10 **99.2** [÷] **32** [Enter]

TI-15 **99.2** [Int÷] **32** [Enter]

12. How do you interpret the result of the following keystrokes to obtain a result for Part **F**?

TI-10 **5** [÷] **36** [Enter]

TI-15 **5** [Int÷] **36** [Enter]

Answer questions 13 - 16 as a *teacher* of mathematics.

13. Outline the different ways the calculator be used for this activity.

14. Did you immediately try to use the calculator in a trial and error fashion, or did you think about the reversibility process necessary to solve the problem? How do you think your processes would differ from those of children?

15. How would you encourage children to find more than one solution?

16. How would you help children to determine the algebraic representation between the two columns of numbers presented above?

Answers and comments

Questions **1 – 7**

The following pertains only to the investigation in Part **A** of the problem. This problem poses an interesting look at reversibility, the same situation that makes 3 + ___ = 8 difficult for first grade children. An interesting connection to division can also be seen. One solution can be found by using the following keystrokes and interpreting the result.

	Keystroke	Result
TI-10	**177** [÷] **21** [Enter]	**8 r 9**
TI-15	**177** [Int÷] **21** [Enter]	**8 r 9**

This demonstrates that starting at 9 and skip counting by 8 twenty-one times will produce the desired result. Of course, there are multiple solutions as shown in question 7, including starting at 177 and skip counting by 0, or starting at 156 and skip counting by 1. It is important to note the values in question 7 are only those for which both values are non-negative integers. Other possibilities, such as skip counting by -1 or 4.2 would also be possible, as would starting at 140. Similar explanations are in order for parts **B - F**.

Skip counting has different interpretations, but a possible solution to the skip counting problem in Part **A** could have been to start at 177 and skip count by 0 twenty-one times, or to start at 156 and skip count by 1 twenty-one times.

Questions **8 – 9**

In every row in the table, 21 times (*number to skip count by*) plus (*number to start at*) = 177. In this relationship, every decrease by one in the *skip count* number produces an increase of 21 in the *start at* number. To skip count by 6.5, you can determine the starting value by calculating 177 - 21(6.5), or 40.5. Similarly, to start at 17.3, you can determine the skip count by calculating (177 - 17.3) ÷ 21, or (159.7 ÷ 21).

Whatever relationships seen in Part **A** holds for those in **B - F**. That is, a solution for **B**, **C**, and **D** can be found using the following keystrokes.

Part	TI-10 Keystrokes	TI-15 Keystrokes
B	**99** [÷] **15** [Enter]	**99** [Int÷] **15** [Enter]
C	**158** [÷] **22** [Enter]	**158** [Int÷] **22** [Enter]
D	**481** [÷] **19** [Enter]	**481** [Int÷] **19** [Enter]

The following solution is one of many solutions that can be found for each part.

- Part **B**: Start at 9 and skip count by 6
- Part **C**: Start at 4 and skip count by 7
- Part **D**: Start at 6 and skip count by 25

Questions **10 - 12**

Note that the calculator will not produce a solution for part **E** with the following set of keystrokes.

TI-10 **99.2** [÷] **32** [Enter]

TI-15 **99.2** [Int÷] **32** [Enter]

A different technique must be used. One solution can be to skip count by 2 thirty-two times. That will result in a total skip counting of 64. In order to get to 99.2, you would need to start at 35.2. Of course you could use that same logic and skip count by any number you wished.

The calculator will produce a solution for part **F** by using

TI-10 **5** [÷] **36** [Enter]

TI-15 **5** [Int÷] **36** [Enter]

The result, **0 r 5**, means that you can start at 5, skip count by 0 thirty-six times, and obtain a result of 5. There are many possible solutions to the problem. One way to avoid skip counting by 0 is by thinking: "If I skip counted by 2 thirty-six times, I would have a total skip counting of 72. If I am only now at 5, I must have started at -67 (5 - 72)." A second solution can be obtained by thinking: “If I skip counted by 3 thirty-six times, I would have a total skip counting of 108. If I am only now at 5, I must have started at –103(5 – 108).” Numerous other solutions can be found in a similar manner.

Questions **13 – 16**

One of the most difficult situations to address in this situation is the need for using the calculator for more than a trial and error fashion. Students should understand that they could have “skip counted by” any number, although the number chosen would affect the “start by” number. The use of a table to record possible “start at” and “skip count by” values should lead one to conclude that a linear relationship exists, and from this relationship the algebraic representation can be determined.

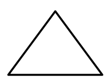
Using calculators to assess patterns, functions, and algebraic reasoning

A calculator can provide many advantages when investigating patterns. It can be used to quickly generate numerous terms in a pattern, to find a value for a term such as the 88th term, to provide a visual graph that indicates how a pattern may be growing, or to find an equation for a pattern. Each of these uses allows students to explore patterns, functions, and algebraic reasoning in more depth that can be quickly done without the use of a calculator. If needed, physical materials should be made available. The number of items asked will depend on the grade level involved. Two assessment items for exploring patterns follow. Of course, not every question should be asked of every child. These, however, do provide a wide range of questions that assess some aspects of algebraic thinking.

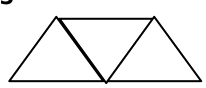
Assessment Item 1

(Adapted from *Stadium Walls*, Texas Instruments, 1998.) An architect designs roof trusses for a building with the following design. These trusses are made from steel beams arranged in the form of equilateral triangles. Each line segment (each side of the triangle) represents a steel beam.

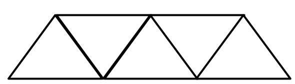
Truss of length 1



Truss of length 2



Truss of length 3



Use toothpicks or other materials to investigate the following questions. Make a table with your findings.

a. How many beams are needed for a truss of length 7?

b. How many beams are needed for truss of length 10?

c. How many beams are needed for a truss of length 23? Of length *n*?

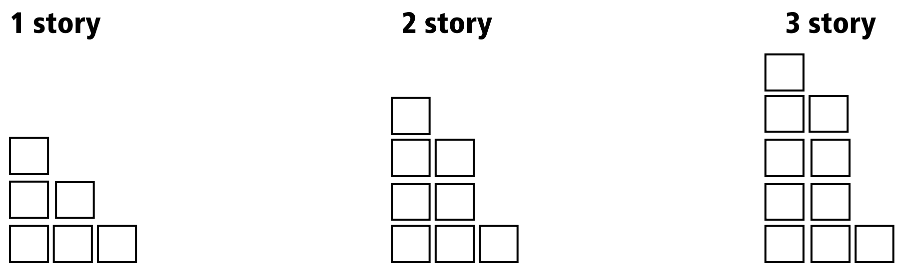
Answers and comments

As the truss length increased by 1, the number of segments increases by 4. For a truss of length 7, one way to see the solution is 3 + 6 x 4 = 27 beams. Similarly, a truss length of 10 needs 3 + 9 x 4 = 39 beams. In general, a truss of length *n* has [3 + (*n* - 1) x 4] beams. This expression can also be written as 4*n* - 1.

Assessment Item 2

(Adapted from *The Twin's Towers*, Texas Instruments, 1998.)

Consider the towers below.



a. What is a recursive pattern that would generate the pattern for the number of blocks needed for the towers?

b. How many blocks are needed for an 8-story tower?

c. How many blocks are needed for a 99-story tower? A 150-story tower?

d. If 4,000 blocks were used, how many stories would be built?

e. How many blocks are needed for an *n*-story tower?

Answers and comments

Recursively, this pattern adds two blocks from one story to the next story. For an 8-story tower, one way to see the solution is [6 + 7 x 2 = 20] blocks. In general, for an *n*-story tower, there are [6 + (*n* - 1) x 2] towers. This expression can also be written as 2*n* + 4. This equation could be obtained by entering the data into two lists and using the linear regression function as part of the STAT capabilities on the TI-73.

Activity overviews for K-6: Patterns, functions, and algebraic reasoning

The following list contains brief descriptions of elementary school student activities that use the calculator as a recording or exploring device for developing understanding of patterns, functions, and algebraic reasoning. The activities can be found on the CD that accompanies this text. Although these activities may involve a calculator that can perform calculations such as linear regression, that is not the emphasis suggested for use by children. Each of the activities are presented in a concrete fashion from which patterns can be used to analyze the investigation. A teacher should work through the activity to the appropriate level for the students involved. Additional activities for primary children can also be found in the references.

Stadium Walls (*Using the TI-73: A Guide for Teachers*, Texas Instruments, 1998.)

Students can use the constant key to help generate the pattern necessary to find the number of beams for walls of lengths 1 to 10. With the TI-73 the values generated could be entered into lists and a linear regression used to determine the $y = ax + b$ equation.

The Twin's Towers (*Using the TI-73: A Guide for Teachers*, Texas Instruments, 1998.)

Students can either use the suggested keystrokes for the TI-73 or determine how to use the operator functions on the TI-10 or TI-15 to generate the patterns desired. This problem provides an interesting look at how to use the TI-73 to generate ordered pairs to represent the steps in a table. You may wish to use only the constant key or operator function to focus on the recursive nature of the growth of the buildings.

A Very Famous Numerical Pattern by Leonardo de Pisa (Mankus, Margo and Klespis, Mark, eds. *Patterns, Patterns, Patterns - Patterns Everywhere!, Math Teacher Educator Short Course for College Professors Teaching Pre-service Teachers with an Elementary School Math Focus*, Technology College Short Course Program, 1999.)

Students can investigate and extend numerical patterns similar to the Fibonacci sequence by using the CONS or OP features. The TI-73 can be used to present a visual representation of the patterns.

Let's Play Video Games (Mankus, Margo and Klespis, Mark, eds. *Patterns, Patterns, Patterns - Patterns Everywhere!, Math Teacher Educator Short Course for College Professors Teaching Pre-service Teachers with an Elementary School Math Focus*, Technology College Short Course Program, 1999.)

Students can use a variety of calculators to examine the patterns generated when determining how many different ways you can buy tokens to play video games at an arcade.

How Totally Square, Part I and Part II (Nast, Melissa, editor. *Discovering Mathematics with the TI-73: Activities for Grades 5 & 6*, Texas Instruments, 1998.)

Students make a variety of sizes of squares, discover patterns, and make predictions based on the patterns observed. The graphing, list, and the ability to obtain equations from the regression function on TI-73 can be used in this investigation.