Sums of Powers and Their Derivatives

By John F. Mahoney

Word processors have changed the way students, and people in general, write. They provide spell checkers and grammar checkers, but more importantly allow writers to revise, add, and subtract anywhere in the document on the fly. The TI-89's and TI-92's Computer Algebra System provides similar tools. It enables users to rely on its algebraic accuracy as well as its numerical accuracy. It enables users to experiment — to cut and paste — without the drudgery of having to work through the details again and again. It allows users to concentrate on setting up problems, rather than on the details of solving each of them. The Computer Algebra System encourages us to explore. Here is an example:

Students regularly are asked to prove statements about the sum of powers of the first *n* natural numbers by mathematical induction. The TI-89 and TI-92 calculators can quickly generate these formulas in both factored form and in expanded form. Consider the following expressions of the sums of powers of the first *n* natural numbers in Figure 1 below.

Do you notice any patterns? I thought I did. It appears as if each sum is essentially the derivative of the next sum. When I investigated this conjecture, I found that I had to first introduce a factor based on the power. Figure 2 shows that

the derivative of $\frac{1}{3}\sum_{i=1}^{n} i^3$ is exactly $\sum_{i=1}^{n} i^2$

the derivative of $\frac{1}{5}\sum_{i=1}^{n} i^5$ is exactly $\sum_{i=1}^{n} i^4$

and the derivative of $\frac{1}{7}\sum_{i=1}^{n} i^7$ is exactly $\sum_{i=1}^{n} i^6$

The other powers *almost* fit this pattern:

The derivative of $\frac{1}{2}\sum_{i=1}^{n} i^2$ is $\sum_{i=1}^{n} i^1 + \frac{1}{12}$

the derivative of $\frac{1}{4}\sum_{i=1}^{n} i^4$ is $\sum_{i=1}^{n} i^3 - \frac{1}{120}$

and the derivative of $\frac{1}{6}\sum_{i=1}^{n} i^6$ is $\sum_{i=1}^{n} i^5 + \frac{1}{252}$

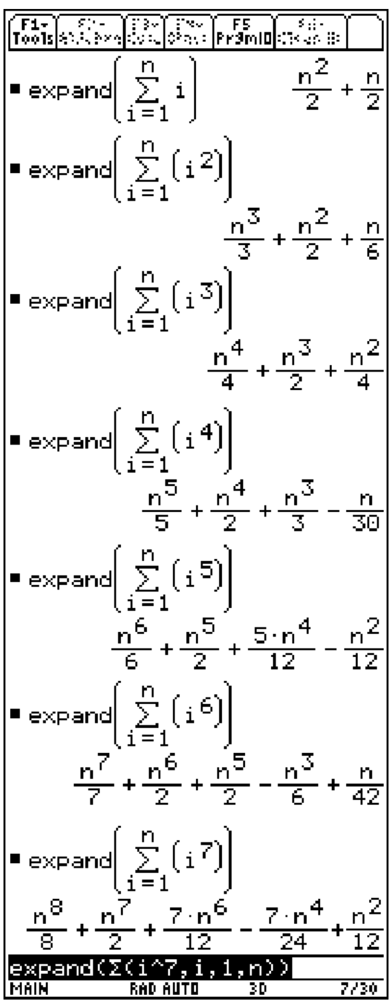

Figure 1

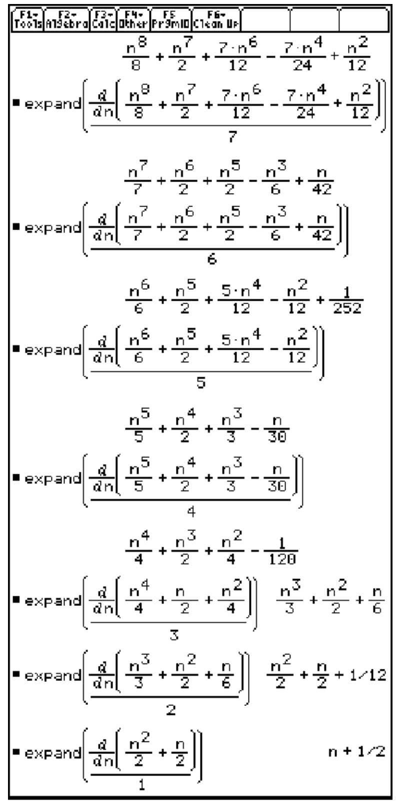

Figure 2

My conjecture "worked" for even powers, and "almost worked" for odd powers. Why did it work? I didn't know. Where did the numbers 1/12, -1/120, 1/252 come from? I had no idea, either. I did some more investigation and found that the next such numbers are -1/240, 1/132, -691/32760, 1/12, -3617/8160, 43867/14364, and -174611/6600.

I decided to try looking up the numbers on the Web. The On-Line Encyclopedia of Integer Sequences **(http://www.research.att.com/~njas/sequences/)** identifies the sequence of denominators of these numbers as coming from the "asymptotic expansion of harmonic numbers." But I didn't know what that phrase meant, so I tried, as a long shot, looking up the sequence of numerators: 1, -1, 1, -1, 1, -691, 1, -3617, ... I lucked out and found that my numerators were essentially the numerators of the Bernoulli[1] numbers. This led me to research Bernoulli numbers and I learned that the Swiss mathematician Jacob Bernoulli (1654 - 1705) had derived a formula for the sum of the integral positive powers of the first *n* natural numbers. A quick consequence of his formula is that the derivative, with respect

to *n*, of $\frac{1}{p}\sum_{i=1}^{n} i^p$ is $\sum_{i=1}^{n} i^{p-1} + \frac{B_p}{p}$

where B_p is the p^{th} Bernoulli number. The first 6 Bernoulli numbers are 1/2, 1/6, 0, -1/30, 0, 1/42. All the odd Bernoulli numbers, after the first, are equal to zero which explains why my conjecture worked so nicely with the odd examples.

In the end all I did was rediscover a result which had been known and proven hundreds of years ago. But I learned a lot and found my interest in one aspect of mathematics reawakened. I am in awe of Jacob Bernoulli's work and elegant proof. Imagine what he would have been able to do with the aid of a TI-89 or TI-92!

Notes: This article was inspired by a talk given by Shin Watanabe, University of Tokai, at the International T^3 conference in 1998. The two figures in the article are the result of cropping, on a word processor, several screen shots from a TI-92.

1) "100 Great Problems of Elementary Mathematics", Heinrich Dorrie, Dover Publications, 1965.