Implicit Differentiation

ID: 8970

Time required
45 minutes

Activity Overview

*Students will be introduced to the concept of implicit differentiation. They will begin by solving a relation for y and use methods with which they are already familiar to find the derivative of the relation. Next, they will use the **impDif** command to find an alternate form of the derivative, followed by verifying that the two forms are equal. Students will also learn how to perform implicit differentiation by hand, using the **impDif** command to check their results. They will be asked to find the numerical derivative of a relation for specific x-values, and a graphical connection to the results found using implicit differentiation will be made.*

Topic: Derivatives

- *Implicit differentiation*

Teacher Preparation and Notes

This investigation offers an opportunity to introduce the concept of implicit differentiation.

- *Students should be familiar with finding derivatives of functions where y is explicitly defined in terms of x. Students should also be familiar with the Chain Rule.*
- *This activity is designed to be **student-centered** with the teacher acting as a facilitator while students work cooperatively.*
- *Reference the Chain Rule to emphasize that y′ (or $\frac{dy}{dx}$) must accompany taking the derivative of expressions containing y.*
- *Provide students with additional practice finding derivatives using implicit differentiation. Include examples such as $\sin(2x - 7xy) = 16y$ that require using the chain rule, product rule, etc.*
- *Notes for using the TI-Nspire™ Navigator™ System are included throughout the activity. The use of the Navigator System is not necessary for completion of this activity.*
- ***To download the student TI-Nspire document (.tns file) and student worksheet, go to education.ti.com/exchange and enter "8970" in the keyword search box.***

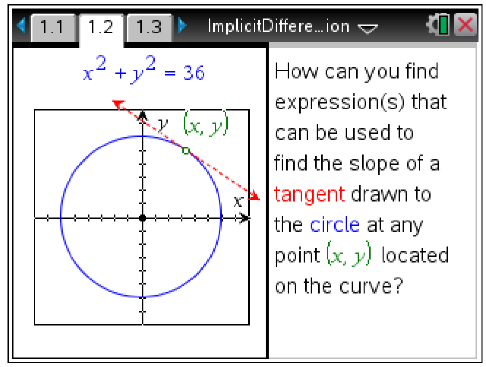
Associated Materials

- *ImplicitDifferentiation_Student.doc*
- *ImplicitDifferentiation.tns*

Suggested Related Activities

To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword search box.

- *Move Those Chains (Chain Rule) (TI-Nspire technology) — 11363*
- *What's the Differential, Dr. Implicit? (TI-Nspire technology) — 11581*



One focus question defines this activity: *How can you find the derivative of a relation, F(x, y), that is not solved for y?*

Use page 1.2 as a discussion point for this focus question.

Students can grab the open point and move the tangent line around the curve. Encourage students to realize that $x^2 + y^2 = 36$ can be solved for y, yielding two equations—one that defines the top semicircle and one that defines the bottom semicircle.

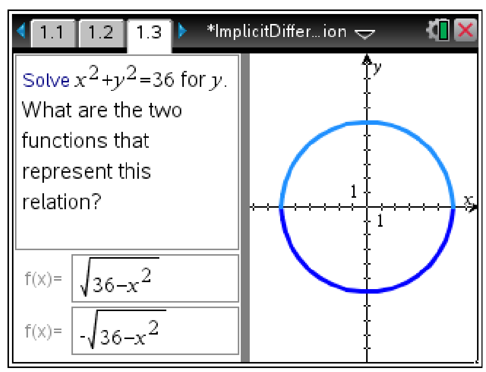
Explain to students that they will in fact solve $x^2 + y^2 = 36$ for y and use the resulting equations to find the derivative of $x^2 + y^2 = 36$.

Also explain that an alternate method for finding the derivative of $x^2 + y^2 = 36$, called *implicit differentiation*, will be used for this example as well as two additional examples.

TI-Nspire Navigator Opportunity: *Quick Poll*

See Note 1 at the end of this lesson.

Problem 1 – Finding the derivative of $x^2 + y^2 = 36$



Step 1: Students will first rewrite $x^2 + y^2 = 36$ as $f_1(x) = \sqrt{36 - x^2}$ and $f_2(x) = -\sqrt{36 - x^2}$.

Step 2: The derivatives of $f_1(x)$ and $f_2(x)$ are:

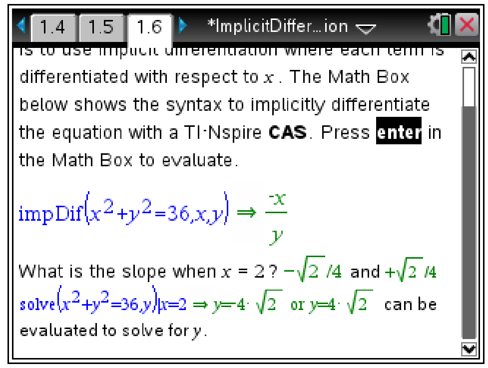
$$\frac{d}{dx} f_1(x) = \frac{-x}{\sqrt{36 - x^2}} \text{ and } \frac{d}{dx} f_2(x) = \frac{x}{\sqrt{36 - x^2}}.$$

Using these derivatives, the slopes of $x^2 + y^2 = 36$ at $x = 2$ are found to be :

$$\frac{-2}{\sqrt{36 - 2^2}} = -\frac{\sqrt{2}}{4} \approx -0.354 \quad \text{and} \quad \frac{-(-2)}{\sqrt{36 - (-2)^2}} = \frac{\sqrt{2}}{4} \approx 0.354 .$$

Students should be encouraged to reference the diagram on page 1.2 to see if these slopes make sense. They should find that the slopes for any value of x should be equal in magnitude but opposite in sign.

Step 3: Next, students will use the **impDif** command (**MENU > Calculus > Implicit Differentiation**), shown at the right, to find an alternative form for the derivative of $x^2 + y^2 = 36$.



Students will use $\frac{dy}{dx} = \frac{-x}{y}$ to find the slopes of the tangents to $x^2 + y^2 = 36$ at $x = 2$. First, they will need to solve for y:

$$(2)^2 + y^2 = 36 \rightarrow y^2 = 32 \rightarrow y = \pm 4\sqrt{2}$$

Substituting $(2,\ 4\sqrt{2})$ and $(2,\ -4\sqrt{2})$ into the formula for the derivative yields

$$\frac{dy}{dx} = \frac{-2}{4\sqrt{2}} = -\frac{\sqrt{2}}{4} \approx -0.354 \quad \text{and} \quad \frac{dy}{dx} = \frac{-2}{-4\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354\,.$$

These results are consistent with those found in Step 2.

Step 4: It can be shown that the derivatives of $f_1(x)$ and $f_2(x)$ that were found earlier are equal to the result found using the **impDif** command by rewriting the formula $\frac{dy}{dx} = \frac{-x}{y}$ strictly in terms of x:

$$\frac{dy}{dx} = \frac{-x}{y} = \frac{-x}{\sqrt{36 - x^2}} \leftarrow \frac{d}{dx}\left(f_1(x)\right) \quad \text{and}$$

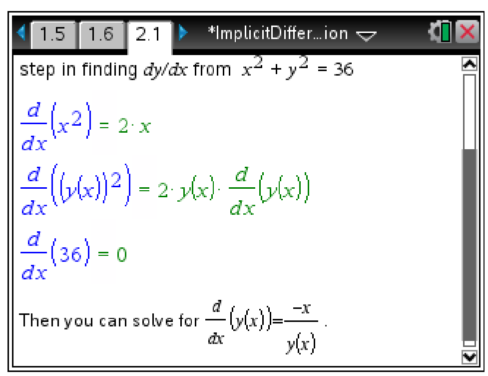
$$\frac{dy}{dx} = \frac{-x}{y} = \frac{-x}{-\sqrt{36 - x^2}} = \frac{x}{\sqrt{36 - x^2}} \leftarrow \frac{d}{dx}\left(f_2(x)\right)$$

TI-Nspire Navigator Opportunity: *Class Capture*

See Note 2 at the end of this lesson.

Problem 2 – Performing implicit differentiation by hand

Step 1: Take the derivative of both sides of $x^2 + y^2 = 36$ with respect to x:



$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(36)$$

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(36)$$

$$2x + \frac{d}{dx}\left(y^2\right) = 0$$

The expression $\frac{d}{dx}\left(y^2\right)$ can be evaluated using the **Derivative** command in the *Notes* application on page 2.1 as shown to the right.

This is a very important step when using implicit differentiation. Explain that $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx} = 2yy'$ and this result can be justified using the chain rule:

$$
\begin{aligned}
y^2 = \left(f(x)\right)^2 \quad \rightarrow \quad \frac{d}{dx}y^2 &= \frac{d}{dx}\left(f(x)\right)^2 \\
&= 2 \cdot f(x) \cdot f'(x) \\
&= 2\,y\,y'
\end{aligned}
$$

Step 2: Students can now finish finding the derivative using implicit differentiation:

$$
\begin{aligned}
\frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(36) \\
\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(36) \\
2x + 2y\frac{dy}{dx} &= 0
\end{aligned}
$$

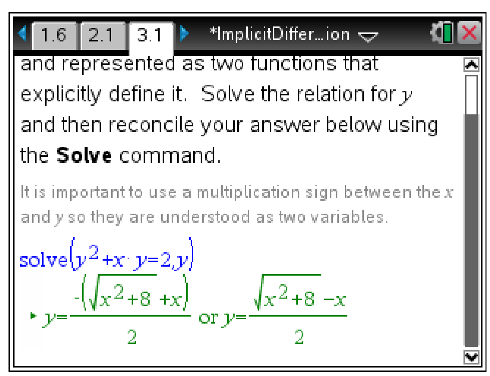
Solving for $\frac{dy}{dx}$:

$$
\begin{aligned}
2x + 2y\frac{dy}{dx} &= 0 \\
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= \frac{-x}{y}
\end{aligned}
$$

This is the same as the result obtained using the **impDif** command.

Problem 3 – Finding the derivative of $y^2 + xy = 2$

Step 1: The relation $y^2 + xy = 2$ can be solved for y by hand by completing the square. The **Solve** command can also be used to find two functions, $f_1(x)$ and $f_2(x)$, that *explicitly* define $y^2 + xy = 2$. (***Note:*** Be sure that students enter the multiplication symbol between the x and y of the term xy. Otherwise, that term is treated as one variable, named ***xy***.)

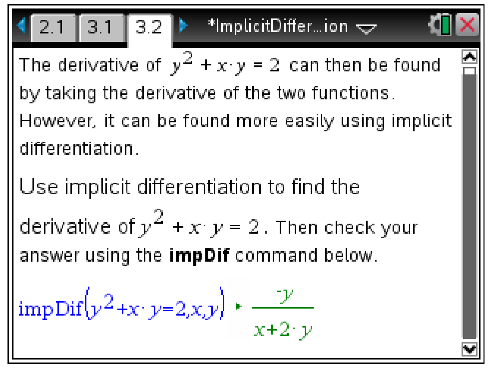


While the derivatives of $f_1(x)$ and $f_2(x)$ can be found using methods learned earlier, it should be stressed that implicit differentiation provides a more convenient method.

TI-Nspire Navigator Opportunity: *Class Capture*

See Note 3 at the end of this lesson.

Step 2: Using implicit differentiation, the derivative of $y^2 + xy = 2$ can be found by hand:



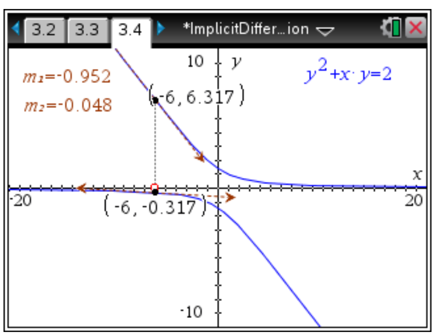
$$\frac{d}{dx}y^2 + \boxed{\frac{d}{dx}xy} = \frac{d}{dx}2$$
$$2yy' + \boxed{xy' + y} = 0$$
$$2yy' + xy' = -y$$
$$(2y + x)y' = -y$$
$$\frac{dy}{dx} = \frac{-y}{2y + x}$$

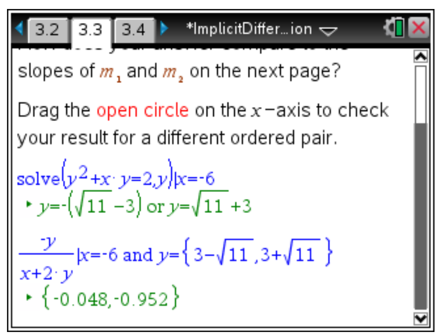
Check with students to ensure they use the product rule when finding the derivative of *xy* (shown in the box above):

$$\frac{d}{dx}(xy) = x \cdot \frac{d}{dx}y + y \cdot \frac{d}{dx}x$$
$$= xy' + y$$

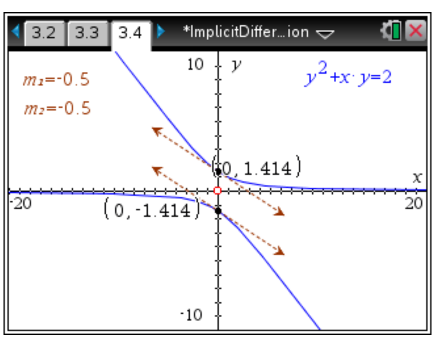
Students will confirm their answer by using the **impDif** command on page 3.2.

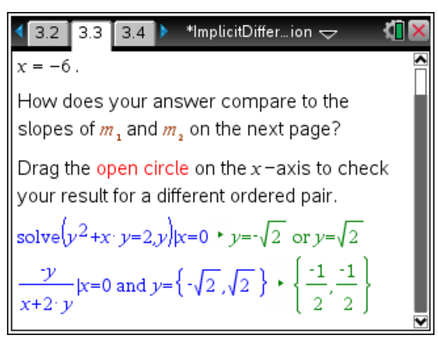
Step 3: This result will also be checked graphically by inspecting tangents to $y^2 + xy = 2$ at two or more points. The screen on the left shows the slopes of the tangents to $y^2 + xy = 2$ at $x = -6$. The screen on the right shows the numerical value of these slopes found by using the formula for the derivative found in Step 2.



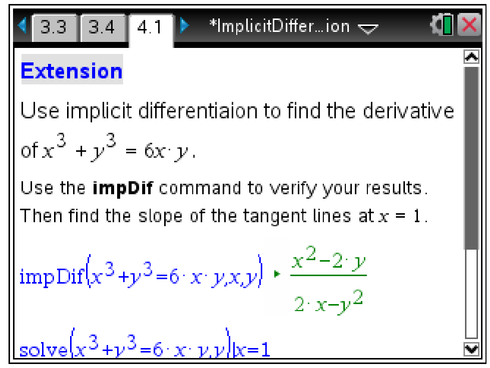


The screens below show a similar analysis for the tangents to the graph at $x = 0$.





Extension – Finding the derivative of $x^3 + y^3 = 6xy$



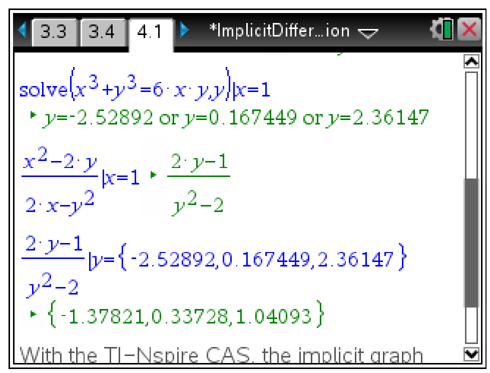
Step 1: The relation $x^3 + y^3 = 6xy$ cannot be solved explicitly for y. In this case, implicit differentiation <u>must</u> be used.

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}6xy$$
$$3x^2 + 3y^2y' = 6xy' + 6y$$
$$3y^2y' - 6xy' = 6y - 3x^2$$
$$\left(3y^2 - 6x\right)y' = 6y - 3x^2$$
$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

The **impDif** command may be used to confirm this result. Students may need to do the work to see the two solutions are the same.

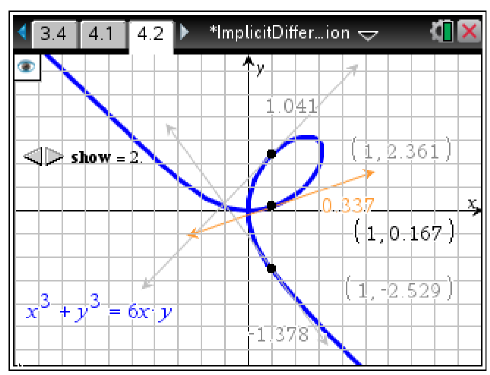
$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x} = \frac{-x^2 + 2y}{-2x + y^2} = \frac{-(x^2 - 2y)}{-(2x - y^2)} = \frac{x^2 - 2y}{2x - y^2}$$

Step 2: Students will use $\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$ to find the slopes of the tangents to $x^3 + y^3 = 6xy$ at $x = 1$. Using the **Solve** command, students will find that the ordered pairs with x-coordinate equal to 1 are (1, –2.529), (1, 0.167) and (1, 2.361).



The slopes of the tangents can now be calculated on a *Calculator* application page or a *Notes* application page as shown here.

For the ordered pair (1, 2.36147), the slope of the tangent is 1.041. For (1, 0.167), the slope is 0.337. The slope for (1,-2.529) is –1.378.



TI-Nspire Navigator Opportunity: ***Quick Poll***

See Note 4 at the end of this lesson.

TI-Nspire Navigator Opportunities

Note 1

Problem 1. *Quick Poll*

Question applications on page 1.3 and 1.4 provide a great opportunity to do Quick Polls. These questions with a graph of the equation showing have the added beneficial feature of showing graphically the equations of the whole class. Click the Graph Data View in the Review Workspace to show students' solutions.

Note 2
Problems 1&2, *Class Capture*
This would be a good place to do a Class Capture to verify students are entering the correct commands to find the answers.

Note 3
Problem 3, *Class Capture*
You may want to use Class Capture to verify students are working through the problem and able to use the commands correctly.

Note 4
Entire Activity, *Quick Poll*
You may choose to use Quick Poll to assess student understanding throughout the lesson. The worksheet questions can be used as a guide for possible questions to ask.