

# HAIRPIN Curves

By Dave Slomer  
Eightysomething! Editor



Big Mountain Road leads from the city of Whitefish, MT, to Big Mountain. The road, like most roads up steep mountains, is made less steep by meandering through a series of switchbacks – “hairpin curves” to we flatlanders. I had been told big buses and semi-tractor-trailer rigs traversed this road through all seasons and began to wonder how big of a vehicle could make it around the sharpest of the switchbacks. After looking at two electronic maps – both using Global Positioning System satellite data – I saw that one turn going up Big Mountain Road might just be as sharp as 150°. This made me wonder if my truck could maneuver my 32-foot fifth-wheel trailer up that sucker!

As is usual in applications, I made some assumptions:

- The approaches on either side of each switchback were straight.
- The sides of the road segments were parallel.
- There was a distinct point that was the vertex of the angle so formed.

I saw no reason to assume the widths of the individual segments of roadway were equal on either side of the switchback, since I was sure I was going to enlist the power of the TI-92 Plus to help. Figure 1 shows a generalized road in which the angle of the bend is  $\theta$  (that is, the turn is  $180-\theta$ ). (A text file that issues the commands for going through this article is stored at: <http://homepages.uc.edu/~slomerdd/>.)

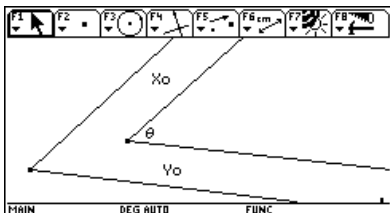


Figure 1 – General hairpin curve angle of  $\theta$  with roadway segment widths  $x_0$  and  $y_0$ . The trailer (heavy line segment of length  $L$  in figure 2) will barely fit around this switchback.

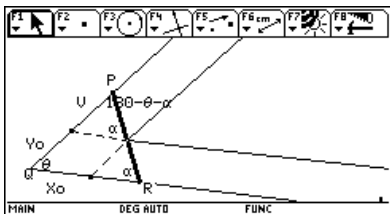


Figure 2 – The sketch from which all the math below comes. Extending the inside edges of each road segment to the other side of the other segment forms at least three similar triangles.

Similar triangles and the Law of Sines provide all the facts necessary to set up the problem:

In the “big” triangle: 
$$\frac{\sin \theta_0}{l} = \frac{\sin \alpha}{v+y_0}$$

In the “upper small” triangle: 
$$\frac{\sin \alpha}{v} = \frac{\sin (180-\alpha-\theta_0)}{x_0}$$

(I have included “subscripts” of zero on  $\theta$ ,  $x$  and  $y$  to emphasize that they are unspecified constants (parameters) representing the dimensions of any given hairpin curve.)

A similar, more trivial, calculus problem involves a pipe being moved around a corner where  $\theta = 90^\circ$ . The intuitive leap, that the *longest* pipe that will go around the corner is the *shortest* one for which the pipe touches both outer walls and the corner, is key to the solution both in that case and here. So we need to minimize  $l$ , a function of two variables,  $\alpha$  and  $v$ . The algebraic preliminaries end just before figure 5, where the calculus begins.

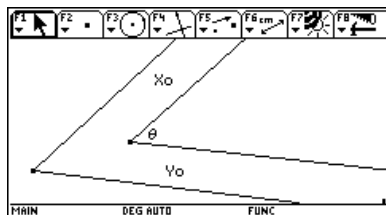


Figure 3 – Removing  $v$  from the picture by solving the second equation for  $v$  and substituting into the first equation.

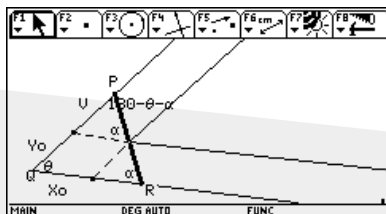


Figure 4 – Solving for  $l$ , the length.

The following commands, which were actually issued (by the text file referred to earlier) immediately after

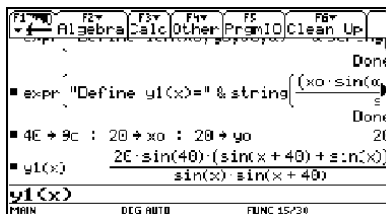
solving for  $l$ , define function  $len$ , whose value depends on variable  $\theta$  and parameters  $x_0$ ,  $y_0$  and  $\theta_0$ .

```
right(ans(1))
expr("Define len(x_0,y_0,theta,alpha)=" & string(ans(1)))
expr("Define y1(x)=" & string(ans(2))|alpha=x)
```

Those commands define  $len(x_0,y_0,\theta_0,\alpha)$  to be  $(x_0 \cdot \sin(\alpha) + y_0 \cdot \sin(\alpha + \theta_0)) \cdot \sin(\theta_0) / (\sin(\alpha) \cdot \sin(\alpha + \theta_0))$

and  $y_1(x)$  to be  $\sin(\theta_0) \cdot (y_0 \cdot \sin(x + \theta_0) + x_0 \cdot \sin(x)) / (\sin(x) \cdot \sin(x + \theta_0))$

It is necessary to give specific values to the parameters before asking the TI-92 to solve for  $\alpha$ . To look at a graph of  $y_1$  or a table of its values,  $\alpha$  must be changed to  $x$ . If we assume that the road segments are only 20 feet wide and that a 140° turn ( $\theta=40^\circ$ , 40° short of a 180° turn) will need to be made, then  $y_1(x)$  is (see figure below). The angles in the denominator, as



well as those in the similar triangles, suggest that we limit  $x$  to the interval  $(0,140)$ , or  $(0,180-\theta_0)$ , in general.

We now look to calculus for help in solving the problem, a typical optimization problem. The derivative is ugly, no matter how viewed:

$$-\sin(40) \cdot \pi \cdot ((\sin(x))^2 \cdot \cos(x+40) + (\sin(x+40))^2 \cdot \cos(x)) / (9 \cdot (\sin(x))^2 \cdot (\sin(x+40))^2)$$

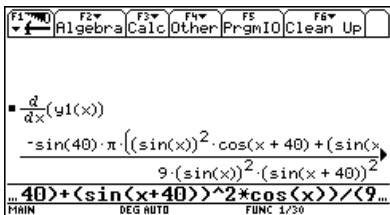


Figure 5 – The derivative of the length function, just stored into y1 as a function of x ( $\alpha$ ) only. The command line has been scrolled to the right to try to reveal what is missing from the numerator.

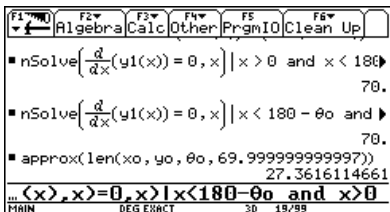


Figure 6 – Setting the derivative to zero and solving numerically. The derivative is zero at  $\alpha=70^\circ$ , which leads to a 27.3-foot maximum trailer length. (The tail end of the nSolve line is in the command line.)

A look at the graph of y1 (see figure 7) suggests that 75 is the only value of  $\alpha$  between 0 and  $180-\theta$  that minimizes len for the given values of the parameters. Presumably this will happen for all choices of road segment widths and hairpin angles.

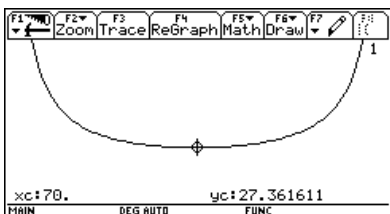


Figure 7 – A graph of the len function near its minimum.

Figures 5 through 7 show that having  $\theta=40^\circ$  gives  $\alpha=70^\circ$ , which yields a point at which the derivative of len is zero. If we just look at the on-screen algebra, knowing that nSolve does not give all solutions to an equation, we might wonder if another might exist, but the graph virtually guarantees not. A 27.3-foot rod would barely make the  $140^\circ$  turn at the corner of a road whose segment widths are 25 feet each. If my 32-foot trailer is going to make it up Big Mountain Road, the roads had better be much wider or the turn much less severe!

Note that the given  $\theta_0$  ( $40^\circ$ ) and the value found earlier for  $\alpha$  ( $70^\circ$ ) satisfy the equation  $2\alpha+\theta_0=180$ . This appears to be true for all roads with equal segment widths. I will use this fact now (it can be easily proved; see the text file/article at <http://homepages.uc.edu/~slomerdd/>), which allows us to solve the equation  $len=32$  for road width, given road angle ( $\theta$ ), or to solve for road angle, given road width.

In figure 8, I try to see ballpark values that WILL permit my trailer to round the hairpin. The first nSolve line shows that, if the road segments are 30 feet wide (including shoulders??), my trailer will pass, even if the hairpin angle is about  $30^\circ$ . In the last line, if the hairpin angle is  $50^\circ$  (and  $\alpha=65^\circ$ ), a 19-foot-wide road will almost work for my 32-foot trailer. (The command line shows the missing part of the first nSolve line.)

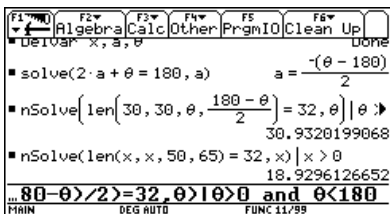


Figure 8 – Wishful thinking?

But I'm worried that  $\theta$  is less than  $40^\circ$  — that the hairpin is even sharper — and that the road width is less than 20

feet! As suggested in figure 8, there are two variables involved in the length problem if the road segments are equal: road width ( $x$ ) and hairpin angle ( $y$ , heretofore called  $\theta$ ). As shown earlier, if the road is 30 feet wide, I will have no problem getting my trailer up the road, so call that  $x_{max}$ . Let  $y$  go up to  $90^\circ$ , although, as shown, values beyond  $50^\circ$  are of no worry — No need to use the time-consuming ZoomFit. Clearly  $z_1$  will have a maximum at  $(x_{max}, y_{max})$ , since the widest, least sharp road allows maximum lengths.

Under the conditions,  $z_1$  (trailer length) is rather simple (see figure 9). Storing 0 to ncontour turns off contour level graphing, after which DrwCtour 32 asks for the one  $z$ -level of interest — my 32-foot trailer. The cursor in the graph of  $z_1$  (fig. 10) tells me that a 27-foot-wide road with a  $35^\circ$  hairpin angle will barely accommodate a 32-foot trailer. Clearly, as  $x$  (road width) decreases,  $y$  (hairpin angle) had better increase, and does. The dark curve, a contour, shows all points at which  $z=32$ , but there is not a lot of directly useful information.

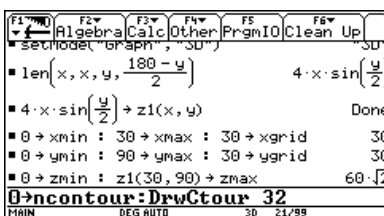


Figure 9 – Preparing for 3D graphing of function len.

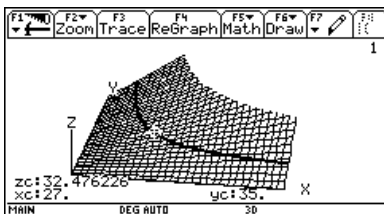


Figure 10 – The len surface with a contour for  $z=32$ .

Since  $z_1(x,y)=4x(\sin(y/2))$ , the contour in the graph is the set of points for which  $4x(\sin(y/2))=32$ .

The situation might be better viewed with a 2D graph, possibly of the inverse of that relation —  $x=8/\sin(x/2)$ . Figures 11 through 13 show 2D aspects of the problem, as does the contour in figure 10.

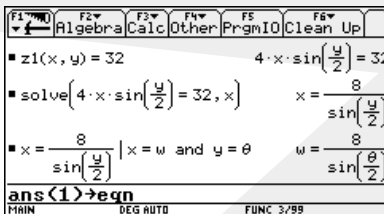


Figure 11 – The problem situation in two variables. The command line stores the last equation into the Solver.

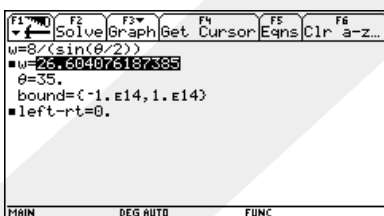


Figure 12 – The TI-92 Plus Solver, as set up in figure 11. Here we see that, if  $\theta=35^\circ$ , the road width had better be well over 26 feet.

In the Solver (fig.12), we can play with  $w$  and  $\theta$  interactively, but a better numeric way than the Solver's one-at-a-time results may be afforded by a look at the usual eight-at-a-time results that a table produces. So we store  $8/\sin(x/2)$  into  $y_1$  and look at a table of ordered pairs [in the form (road width, hairpin angle)]; see fig. 13] that will accommodate a 32-foot trailer. This is more useful than a graph and is the best view of the problem so far. It shows that (for example), if the road width is 21 feet,  $\theta$  had better not be much less than  $44^\circ$ ; if  $\theta$  drops to  $38.5^\circ$ , the road needs to be 24 feet wide.

But are the data in figure 14 the “ultimate” view of the problem? Highlighted is evidence that my trailer will not quite pass a 35° hairpin with segments only 26 feet in width. Nearby we see that if  $\theta=40^\circ$ , I’m alright with a 26-foot road, and if the road is 28 feet, I’m okay with a 35° hairpin.

x	y2
12.	76.534177868
15.	61.290380604
18.	51.139625772
21.	43.899234126
24.	38.477874758
27.	34.269260558
30.	30.909626441
33.	28.167492177

Figure 13—Table in which  $x$  contains a road width that determines  $y_2$  as the maximum corresponding road angle ( $\theta$ ).

DATA	road	$\theta=20$	$\theta=25$	$\theta=30$	$\theta=35$	$\theta=40$	$\theta=45$
	c1	c2	c3	c4	c5	c6	c7
2	16	11.1	13.9	16.6	19.2	21.9	24.5
3	18	12.5	15.6	18.6	21.7	24.6	27.6
4	20	13.9	17.3	20.7	24.1	27.4	30.6
5	22	15.3	19.	22.8	26.5	30.1	33.7
6	24	16.7	20.8	24.8	28.9	32.8	36.7
7	26	18.1	22.5	26.9	31.3	35.6	39.8
8	28	19.4	24.2	29.	33.7	38.3	42.9

Figure 14— A “3D table” of the problem situation.

The following text file produced the data variable in figure 14:

```
C:seq(w,w,14,30,2)»widthlst
C:"NewData hairpin,widthlst"»cmd
C:for  $\theta$ ,20,45,5:cmd & ",z1(widthlst,1.*" & string( $\theta$ )
& ")»cmd:endfor
C:expr(cmd)
```

The value stored into cmd via repeated concatenation (&) in the text file was:

```
"NewData hairpin,widthlst,z1(widthlst,1.*20),
z1(widthlst,1.*25),z1(widthlst,1.*30),z1(widthlst,1.*35),
z1(widthlst,1.*40),z1(widthlst,1.*45)"
```

(The ‘1.’ portion of each reference to z1 causes *approximate* values to be stored into “NewData” variable hairpin. The headings were entered by hand.)

### Extensions

- My trailer is not a rod, but an 8-foot by 32-foot rectangle (projected onto the plane of the road), so: what to do? Its diagonal is 33 feet. Do we just call its “effective length” 33 and use the results above? This sounds reasonable at first, but what if a tree were right at the vertex at the inside of the hairpin? And what if there were a wall or fence right at the edges of the road? It would seem that some part of the trailer would be in jeopardy for certain hairpins consistent with this assumption.
- The assumption that the hairpin “curve” is formed by line segments is severely limiting in the analysis of the problem — My rig did easily make it up the mountain despite the dire warnings in the article. In reality, the curve *is* a curve — but which one: parabola? circle? something else (no doubt)? Can the analysis be done better with a curve, rather than line segments, as road boundaries? Probably, but how?

### Proof?

Here is what I will call a “convincing calculator argument” that the minimum value of the length for hairpins with equal segment widths occurs when  $\alpha = \frac{180 - \theta}{2}$ .

(It is interesting to note that this occurs when  $\theta + 2\alpha = 180$  or when angle P equals  $\alpha$ ; that is, when triangle PQR is isosceles. It is also true that those equations are equivalent to  $\theta + \alpha = 180 - \alpha$ , which says that the exterior angle at P is supplementary to R in PQR.)

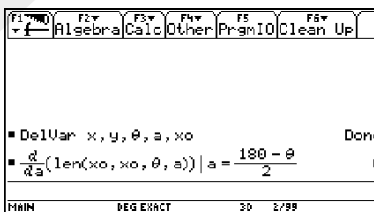


Figure 15 — The derivative of the len function with equal road segment widths at  $a=(180-\theta)/2$  is zero...

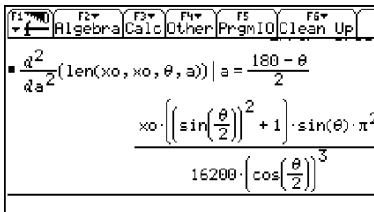


Figure 16 — ... and the second derivative is clearly positive (for values of  $\theta$  between 0 and 180) at the angle in question, PROVING that  $a=(180-\theta)/2$  is sufficient for a minimum for len.

That hardly seems like work, let alone a proof. How much information must be written, by whom and on what medium for us to accept the proof? But what follows is definitely not a proof, just friendly arguing and graphical cajoling, hoping to make the reader “buy” the necessity of having the absolute minimum at the angle in question.

There is no way to view the derivative of len on the TI-92 Plus screen. It is:

$$-x_0 \cdot \pi \cdot ((\sin(x))^2 \cdot \cos(x+y) + \cos(x) \cdot (\sin(x+y))^2) \cdot \sin(y) / (180 \cdot (\sin(x))^2 \cdot (\sin(x+y))^2)$$

The command `getNum(ans(1))/(- $\pi$ *x*sin(y))` discards the denominator (legal, since it has nothing to do with when the numerator is 0) and discards the factors of the numerator that are 0 only at 0. This leaves  $((\sin(x))^2 \cdot \cos(x+y) + \cos(x) \cdot (\sin(x+y))^2)$ , which we store to z1. We will hope to see the significant (x,y) at which z1=0 by drawing a contour for z=0.

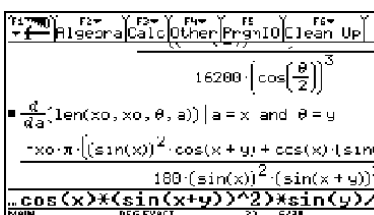


Figure 17— First, express the derivative of len in terms of  $x$  and  $y$  instead of  $a$  and  $\theta$ .

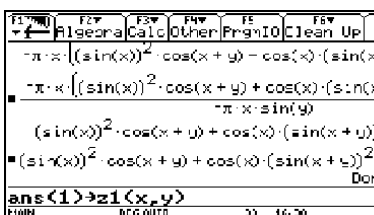


Figure 18— Setting up z1.

The text file below will set a good window in which to graph  $z_1$ , as shown in figure 19.

```
C:setMode("Graph","3D")
C:setGraph("Coordinates","RECT")
C:setGraph("Labels","ON")
C:ZoomFit
C:0»ncontour
C:DrwCtour 0
C:setGraph("Style","WIRE AND CONTOUR")
C:40»eyeθ:20»eyeφ:0»eyeψ
C:0»xmin:90»xmax:15»xgrid
C:0»ymin:90»ymax:15»ygrid
C:setGraph("Axes","BOX")
C:Trace
```

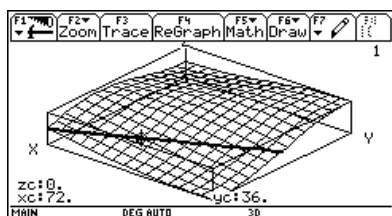


Figure 19 — A picture that is somewhat convincing that the only place  $z_c=0$  occurs is when  $2x_c+y_c=180$ .

During tracing, note that any time you reach  $z_c=0$ , it will be true that  $2x_c+y_c=180$  — in other words (symbols),  $2\alpha+\theta=180$ . In the graph above,  $z_c=0$  when  $x_c=72$  and  $y_c=36$ , since  $2\cdot 72+36=180$ . To reach other points where  $z_c=0$ , use the cursor keys, and aim for the thick line — the contour for  $z=0$ , drawn via the `DrwCtour 0` command in the text file. Note that the thick line appears to be a *line* — the line  $2x+y=180$  in the plane (contour) where  $z=0$ .

Also note that once at a point where  $z_c=0$ , getting to any other point where  $z_c=0$  in the window is done by incrementing  $x_c$  and decrementing  $y_c$  twice (or decrementing  $x_c$  and incrementing  $y_c$  twice) — again showing that  $2x_c+y_c=180$  whenever  $z_c=0$ , in addition to showing that the slope of that line is  $-2$ .

Noting that nowhere else in the window (in which  $x$  and  $y$  range from  $0$  to  $90^\circ$ ) does it seem likely that  $z=0$ , it would seem clear that, not only "IF", but also "ONLY if"  $2\alpha+\theta=180$  will the derivative of  $\ln$  be zero.

## Meet the Editor

TI is proud to announce Julie B. Ealy as the new science editor of **Eightysomething!** Dr. Ealy received her BA from the State University of New York at Buffalo, MS from Northern Illinois University and MPhil and Ph.D. in chemical education from Columbia University.

For 24 years, Dr. Ealy taught all science courses from earth science to physics at the secondary level, including honors and Advanced Placement chemistry in public and private schools. She then pursued a Ph.D. under the guidance of Leonard Fine. She evaluated the integration of molecular modeling into first-year college chemistry and found that molecular modeling both qualitatively and quantitatively makes a difference in students' learning at the atomic and molecular level. She developed self-contained demonstration kits for Flinn Scientific Company and has co-authored two books on chemical demonstrations published by the American Chemical Society; the most recent is *Visualizing Chemistry*.

Dr. Ealy is a consultant for the College Board in AP Chemistry and for Harcourt College Publishers. She has performed research in organometallic synthesis with Claude Yoder and educational research with the late Miles Pickering. She has presented numerous workshops on chemical demonstrations to elementary school teachers through college professors at local, national and international conferences. She recently began adaptation of chemical demonstrations for the CBL and graphing calculator system. She is a ChemBio instructor for TI and has presented numerous Discovery workshops. For the last two years, Dr. Ealy has been teaching part-time at Rye Country Day School and Columbia University. Her present research interests pertain to the integration of writing into first-year college chemistry, evaluation of a college chemistry textbook and development of assessment measures for molecular modeling.