Motion of Related Objects—Introducing Related Rates

by Dave Slomer

At 10pm, a student drops a very small ball from an overpass, 50 feet above ground. Thirty feet away is a bright light atop a pole, also 50 feet above ground. (Refer to figure 1.) In figure 2, the **L**ight is at point **L**. The **F**oot of the pole is point **F**. Before it begins to drop, the ball is at point **O**.

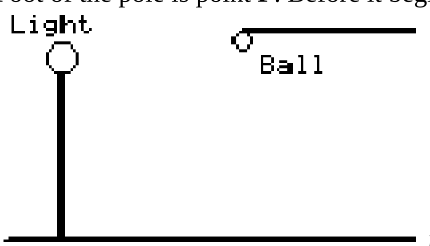


Fig.1

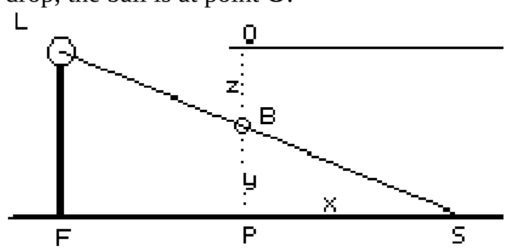


Fig.2

You are going to investigate the relative positions of the ball and its shadow at various times during the ball's fall toward earth. To do so, you need to find an equation somehow relating ball and shadow. At any time after the ball begins to fall (but before it hits the ground), the relationship between its position and its shadow's is determined by identical geometry.

So, consider an arbitrary instant during the time the ball is falling, somewhere near the halfway point of its fall toward earth, such as at point **B** in figure 2. As the ball falls, its shadow moves "from infinity" along the ground toward the point at which it meets the ball. This is point **P,** the "**P**lop" **P**oint, where the ball hits the ground (squashing its shadow). If we consider the ball to be a point, the only light ray of interest is the one (**LS**) that hits the ball, thereby causing the shadow to appear as a point on the ground at point **S**.

As the ball falls, it covers a variable amount of distance, which depends on the time the ball has been falling. The same goes for the shadow. So set **OB** = **z**, **BP** = **y**, and **PS** = **x**.

1. In figure 2, there are two *similar right triangles*. Write a proportion using **x**, **y**, and/or **z**.

2. The ball is freely falling, accelerating at the constant rate of 32 (ft/sec)/sec. Its instantaneous acceleration (***a***) is the derivative (with respect to time) of its instantaneous velocity (***v***). Give an equation for ***v*** in terms of time ***t***. This is easy because ***a*** is <u>*constant*</u>.

   v(t) = ________

   Check: Does $\dfrac{dv}{dt} = -32$ ft/sec^2 ? (Why *negative*?)

3. Keeping in mind that instantaneous velocity (***v***) is the derivative (with respect to time) of the position function (***h***), give an equation for the position of the ball at any time ***t*** (before it plops).

   h(t) = _________

   Check: Does $\dfrac{dh}{dt} = v$?

   Check: Is the initial height 50? (That is, does $h(0) = 50$?)

4. Because of the <u>vertical</u> movement of the ball, your TI-89 will best show the problem situation in **parametric mode**. Graphs defined by parametric equations have their points (x,y) defined by a third variable usually named t, so named because t is so often *time* in the parametric world. So both x and y are functions of t.

   There are 3 things to get into the picture, so there will be 3 parametric equations to define.

   a. The <u>light</u>, which never moves, but should be graphed as a (single) reference point;

   b. the <u>ball</u>, which moves vertically (downward from the overpass); and

c. the <u>shadow</u>, which moves horizontally (leftward from positive ∞).

Let’s put the origin at **F**. Then the coordinates of the light are (0,50) at all times ***t***. It could be “graphed” by making **xt1 = 0** and **yt1 = 50**. Of course, the light doesn’t move, but your ‘89’s “**path**” graph style will make it appear during graphing. How to do so is shown in figure 3.

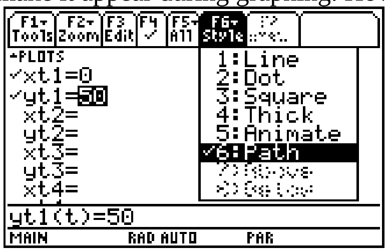


Fig. 3

The ball is moving, but only vertically. Its *x*-coordinate is always 30 (Why?) and its *y*-coordinate is ***h***, where ***h*** is what you got in #3. So, make **xt2 = 30** and **yt2 =** …*whatever you got for its height in #3*…

Although its *y*-coordinate is always 0 (Why?), the *x*-coordinate of the shadow requires you to solve your equation from part 1 for **x**. Note that the *x*-coordinate of the shadow is 30+**x** (Why?). You will have expressed **x** in terms of **y**, so expressing **xt3** in terms of **yt2** is perfectly OK, but it’s hard to type correctly, so you might want to express **xt3** in terms of **t**, as shown in figure 4.

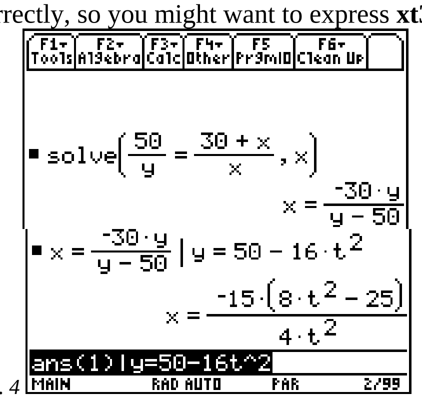


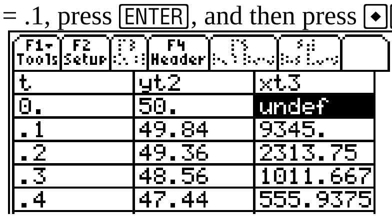
Fig. 4

5. Summary: for your parametric function definitions, you have:

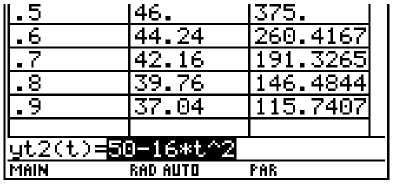
Light:	**xt1** = 0	**yt1** = 50
Ball:	**xt2** = 30	**yt2** = _____
Shadow:	**xt3** = ________________	**yt3** = 0

Put these equations into your ’89. To get into parametric mode, press [MODE][▶][2]. To get the most from the graphs, use **discrete** (**dot**) **graph style**, <u>not connected (line)</u>. To do so, position the cursor on the **yt** (or **xt**) equation for the ball and shadow functions in the function editor ([Y=] screen), press [2nd][F6], and select option 2.

6. Setting the window is always important. A table (fig. 5) can help. It shows that, initially, **S** is “at infinity.” (Why?) But it quickly heads toward **P**, so set **xmax** to 200, ignoring the first .6 seconds of motion. Looking further down through the table suggests setting **tmax** to 1.7 or so (fig. 6), since that is the time just before plop down. (To get this table yourself, press [◆][F4] (**TblSet**), make **tblStart** = 0 and **Δtbl** = .1, press [ENTER], and then press [◆][F5] (**TABLE**).)



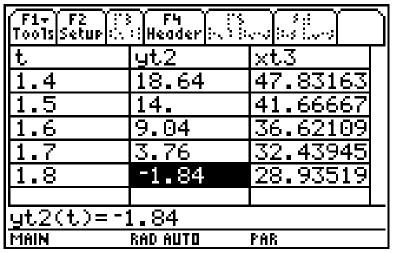
t	yt2	xt3
0.	50.	undef
.1	49.84	9345.
.2	49.36	2313.75
.3	48.56	1011.667
.4	47.44	555.9375



.5	46.	375.
.6	44.24	260.4167
.7	42.16	191.3265
.8	39.76	146.4844
.9	37.04	115.7407

yt2(t)=50-16*t^2
MAIN RAD AUTO PAR

Fig 5



F1 Tools F2 Setup F4 Header

t	yt2	xt3
1.4	18.64	47.83163
1.5	14.	41.66667
1.6	9.04	36.62109
1.7	3.76	32.43945
1.8	-1.84	28.93519

yt2(t)=-1.84
MAIN RAD AUTO PAR

Fig. 6

Since negative time is meaningless in this problem, **tmin** should be 0. To find **tmax** more precisely just think: How long will it take for the ball to hit ground? You'll want to solve **yt2=0**. (Why?)

tmax = _____

7. Keeping in mind that **t** will go from **tmin** to **tmax** by the value of **tstep**, setting **tstep** too small will result in super slow motion, while making it too big will result in super *fast* motion. Make a table of values of **tstep** and corresponding numbers of points that would have to be plotted and base your **tstep** choice on the table data. The relationship you need is $tStep = \dfrac{tMax - tMin}{n}$ where *n* is the number of points that will be plotted for the given **tmin**, **tmax**, and **tstep**.

tstep	*number of points (n)*
10	
1	
.1	
.01	

8. The heights of the poles and location of the ground (on the *x*-axis) suggests making **ymin** = 0,

ymax = 50 and **xmin** = 0, but you might want to make **ymin** negative since it's going to be important to be able to read the trace coordinates when you graph. You might also want to leave space above and to the left of the light pole by making **ymax** larger than 50 and **xmin** negative.

9. Summarize your window: **tmin** = _____ **tmax** = _____ **tstep** = _____

xmin = _____ **xmax** = _____

ymin = _____ **ymax** = _____

10. Graph it! If things are too fast or slow, just change **tstep**. What you should see is what you'd expect—a discrete graph of ball positions and shadow positions for the **t** values dictated by your window. During graphing, press $\boxed{\text{ON}}$ to "freeze" the action as soon as both the ball and the shadow appear in the window. Press $\boxed{\text{ON}}$ again to un-freeze the action. Is the graph consistent with the sketch in figure 2? Does it look like figure 7? The ball should not go "through the ground." The shadow should not pass the plop down point. If things don't look right, retreat and discuss your equations and settings with a

classmate.

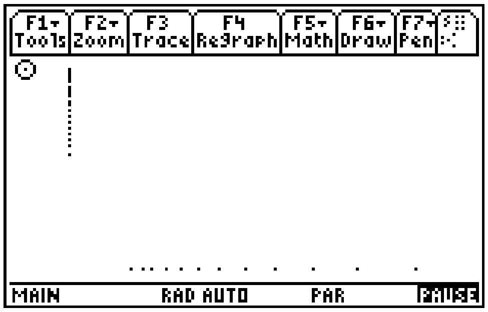


Fig. 7 *(Note the light "shining" at the upper left!)*

11. Once the graphing has finished, trace the ball and shadow graphs so that you can answer the questions in the table below. The trace should resemble figures 8 and 9 below, where we have traced to the positions of ball and shadow at time **t = 1**.

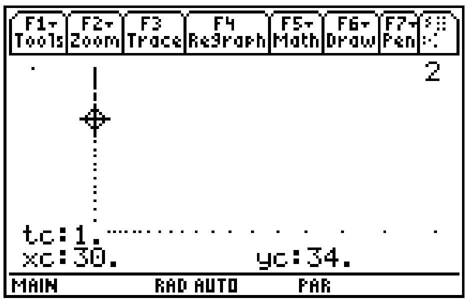


Fig. 8

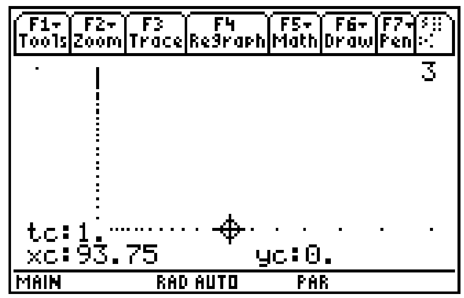


Fig. 9

a.	Which is moving faster initially: ball or shadow?	
b.	Why?	
c.	Which is moving faster at plop-down?	
d.	Why?	
e.	Is there a time when the speeds of the ball and shadow are equal? If so, how might you try to find it? If not, why not?	
f.	Set **tstep** to **0.1** and trace to the instant when **t = 1**. **Estimate** the instantaneous speed of the ball then. ***Show your calculation.*** Keep in mind that each dot represents the position of the ball at the corresponding time.	
g.	Approximately how fast is the shadow moving when **t = 1**? Show your calculation.	
h.	Suppose you wanted to improve on your approximations for the instantaneous speeds. What would you do?	

12. Turn off the light (no joke!) and make a table of just the ball and shadow functions. In the function editor, the F4 key turns off (or on) a particular function. Turn off everything but **yt2** and **xt3**. Set **tblStart** = 1 and **Δtbl** =**.01** and look at the table. What are your approximations now for instantaneous ball speed and shadow speed at time **t = 1**? You might want to use the forward, backward, and symmetric difference quotients.

13. What if you wanted the exact values for the instantaneous speeds? What would you do?

14. As the ball moves, its shadow does so in a predictable way—the objects are related by the laws of physics. But their ***rates*** are also ***related***. The kind of problem you have just experienced is called a ***related rates*** problem. The relationship between the rates can be expressed via the **Chain Rule**.
If *y* = *f*(*t*) *(such as the ball's position)* **and *x* = *g*(t)** *(such as the ball's shadow's position)*, **then their instantaneous rates** *(their derivatives)* **are related by the chain rule**. Fill in the 2 blanks below.

$$\frac{dx}{dt} = \frac{dy}{dt} \cdot \frac{d\_}{d\_}$$

15. To find the "missing" rate in the previous blank, you need the equation from part 1 that relates *y* (the ball's position) to *x* (its shadow's position). Find the derivative of *x* with respect to *y*.

16. Now, use the chain rule to symbolically find the exact speed of the shadow at time $t = 1$.

17. You do not have to use the chain rule to find the shadow's speed if you expressed **xt3** in terms of **t** in part 4. On the $\boxed{\text{HOME}}$ screen, use the symmetric, forward, or backward difference quotient to compute the exact speed at $t = 1$ by letting Δt approach 0.

18. Wasn't this a ball?

Calculus Generic Scope and Sequence Topics:

4. Applications of Derivatives
12. Parametric, Vector and Polar Functions
14. Motion

NCTM Standards: Algebra, Geometry, Problem solving, Connections, Representation