

# Approximate Solutions to Differential Equations – Slope Fields (graphical) and Euler’s Method (numeric)

by Dave Slomer

Leonhard Euler was a great Swiss mathematician. The base of the natural logarithm function,  $e$ , called Euler’s number, is named in his honor. While studying calculus, Euler wondered about graphical and numeric solutions to differential equations, such as

$\frac{dy}{dx} = \frac{y}{2} + x$ , that cannot be solved symbolically through elementary methods such as separation of variables. (Why can it not be so solved?)

In figures 1 through 3 below, you see representations of the differential equation

$\frac{dy}{dx} = \frac{y}{2} + x$ . First, there is a graphical view via a “slope field.” Second, there is a numeric solution, a table of values that “Euler’s method” of solving differential equations yields when the point  $(-3, 2.7)$  is used as a starting point. And third, the connected, piecewise-linear graph of the Euler’s method numeric data is superimposed on the slope field.

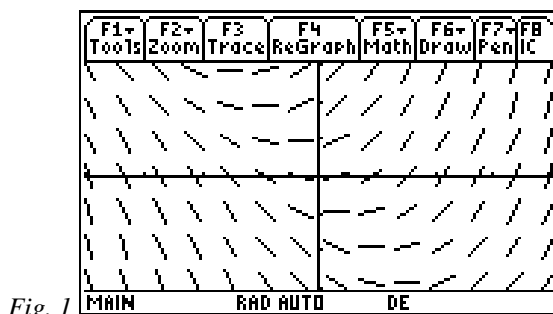


Fig. 1

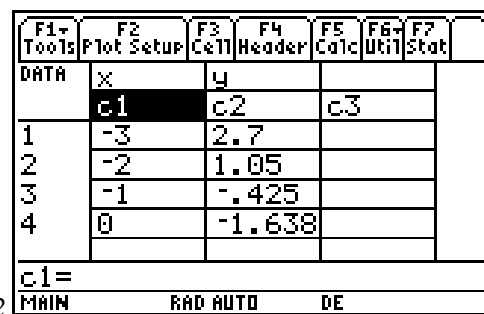


Fig. 2

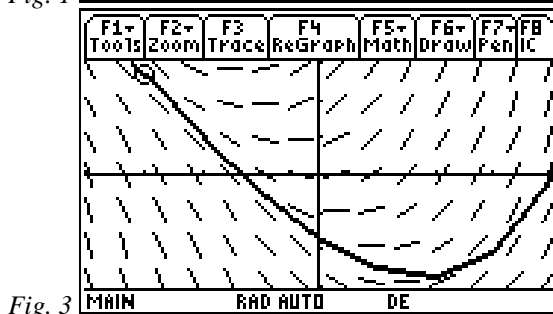


Fig. 3

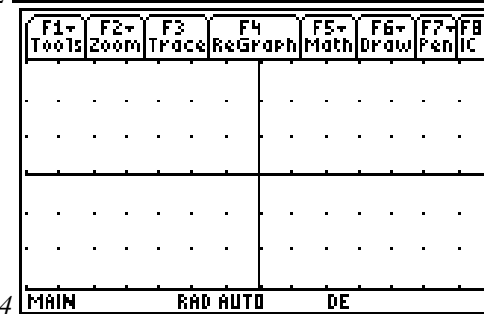


Fig. 4

A slope field is constructed in a “lattice” (or “grid”) of points. See figure 4 for a lattice similar to the one used to produce the slope field in figures 1 and 3. At each point in the lattice, we draw a short line segment whose slope is determined by the differential

equation. For example, at  $(-3, 2)$ ,  $\frac{dy}{dx} = \frac{2}{2} + -3 = -2$ , so a short line segment with slope  $-2$  is drawn, centered at  $(-3, 2)$ .

Euler's method of solving differential equations of the form  $\frac{dy}{dx} = F(x, y)$  (such as the one above) is an interpretation of numeric integration (left-hand Riemann sums), cleverly applied at  $n$  equally-spaced  $x$ -coordinates over an interval  $[a, b]$ . Euler figured that, since the symbols  $dy$  and  $dx$  are related by such relationships as those below, he could easily enough solve the differential equation numerically:

$dx = \Delta x$ , where  $\Delta x = (b-a)/n$ , the length of each subinterval, called the *stepsize* for Euler's method, for good reason, you'll see;

$dy = F(x, y) dx$ , which is essentially the given differential equation; and

$\Delta y \approx \frac{dy}{dx} \cdot \Delta x$ , since all smooth functions look linear over a small enough

interval—that is, since  $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$  if  $\Delta x$  is close enough to 0.

So, Euler's method is this: **pick a point** to start at (the *initial point* or *initial condition*), then **follow the slope field** segment at that point in the direction it points (determined by its slope,  $y'(x, y)$ ) by **moving  $\Delta x$  units horizontally and  $\Delta y$  units vertically, where  $\Delta y \approx y'(x, y) \Delta x$** . Because of how the “rise over run” property of lines works, you'll be at a logical second point (only an approximation, but as close as  $\Delta x$  and  $\Delta y$  let you get), from which the process may be repeated.

By understanding just those thoughts, you can solve differential equations as accurately as you wish (provided  $n$  is big enough and, consequently,  $\Delta x$  is small enough) over some interval  $[a, b]$ . All you need to know is what  $y'(x, y)$  is, and that is just the right-hand side of the differential equation you are working on.

*Exercise 1: Graphically “solve”  $\frac{dy}{dx} = \frac{y}{2} + x$  by making its slope field, by hand, to get a good feel for and understanding of both processes (slope fields and Euler's method).*

In words, the differential equation says “the slope at any point in the slope field is the sum of the  $x$ - and half of the  $y$ -coordinate at that point.” Use a lattice with  $x$  going from  $-3$  to  $3$  and with  $y$  doing the same thing. Start at  $x = -3$ .

$x$	$y$	$dy/dx$	Action to take:
-3	-3	$-3 + -3/2$ , or $-4.5$	Draw short segment with slope $-4.5$ at $(-3, -3)$
-3	-2	$-3 + -2/2$ , or $-4$	
-3	-1		
-3	0		
-3	1		
-3	2		
-3	3		

Finished with  $x = -3$ , we would move to  $x = -2$  and repeat the process for all the  $y$ 's. Then on to  $x = -1, 0, 1, 2$ , and  $3$ , each for all the  $y$ 's. No, thanks!! So, hereafter, we'll let the

TI-89 do it for us. To do so, you have to change the **MODE** for the **Graph** to **6, DIFF EQUATION** (see figure 5).

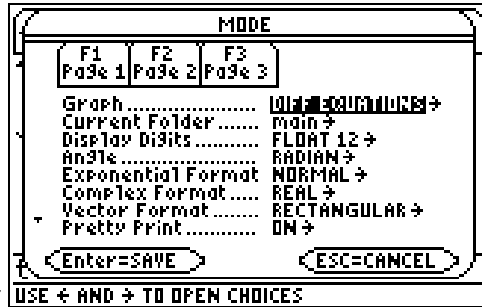


Fig. 5

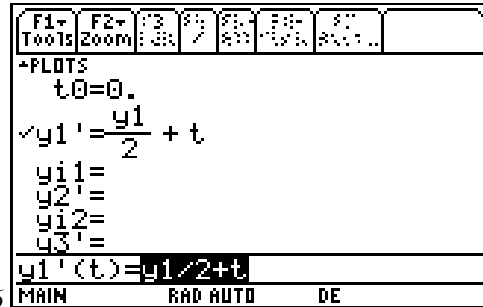


Fig. 6

Then you have to enter the differential equation using **y1** instead of **y** and **t** instead of **x** as shown in figure 6. Setting the **WINDOW** is done pretty much as usual (does it remind you of parametric mode?). To get what's in figure 1, make **x** go from **-4** to **4** and **y** from **-3** to **3**, and set **fdres** to the number of horizontal points to use in the lattice, **14**. (See figure 10.)

Now, apply Euler's method (by hand) with *stepsize* = 1, using (-3,2.9) as the initial point. Fill in the table below. The calculations can be daunting, so feel free to apply the home screen commands shown in figures 7 and 8, which considerably simplify the calculations, although they do involve two complex commands. You do not have to retype the commands over and over. Just use the cursor keys to go up and "grab" the repeated commands (in the right order). Note that **y** changes before **x** does. If you do it right, the last 2 lines of the screen will always be the coordinates of the next point to plot.

Point to plot:		Calculations for next point:		
<i>x</i>	<i>y</i>	<i>dy/dx</i>	$\Delta y, \text{ or } \frac{dy}{dx} \Delta x$	Next <i>y</i> , or <i>y</i> + $\Delta y$
-3	2.9	1.45-3, or -1.55	-1.55 1, or -1.55	2.9-1.55, or 1.35
-3+ $\Delta x$ , or -2	above, 1.35	.675+-2, or -1.325	-1.325 1, or -1.325	1.35-1.325, or .025
-1	.025			
0				
1				
2				
3				

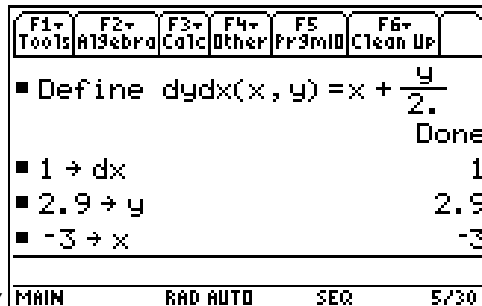


Fig. 7

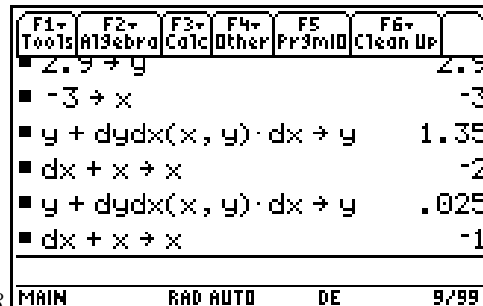


Fig. 8

Finally, plot those points and connect them with line segments right on the slope field. (The graph will look like a smooth enough curve if the stepsize is small enough.) In addition to seeing slope field evidence about how the general solution looks, you will see a graphical representation of a numeric approximation of the solution of the symbolic differential equation  $\frac{dy}{dx} = \frac{y}{2} + x$ , the specific solution that passes through  $(-3, 2.9)$ . You should get what's in figure 3.

It's unlikely you'd ever want to go through that again, so use your '89 hereafter to produce "Euler solutions" to differential equations. You have to specify the initial condition (e.g., that  $y = 2.9$  when  $x = -3$ ) on the **Y** screen (see fig. 9) and set the **WINDOW** (fig. 10). Then you will get what's in figure 3. You can set **ncurves** to any whole number from 1 through 10 to allow drawing solutions for more than one initial condition (see figs. 10 and 12, where 5 solutions have been requested and drawn). Set **tstep** (stepsize) to something smaller than 1 since it's not you that's doing the work—the more points the smaller  $\Delta x$  and the better the accuracy. Consider not using **EULER** as the **Solution Method** (see fig. 11), since the **RK** (Runge-Kutta) method, based on Simpson's rule for numerically approximating integrals, is more accurate.

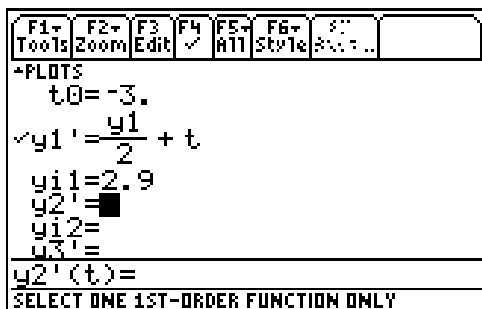


Fig. 9

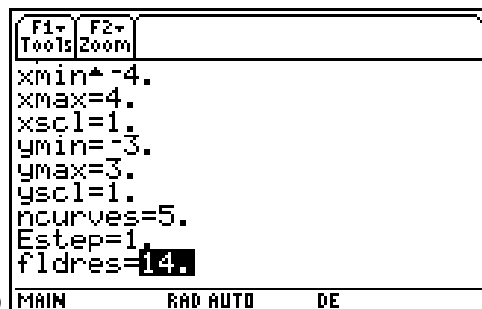


Fig. 10

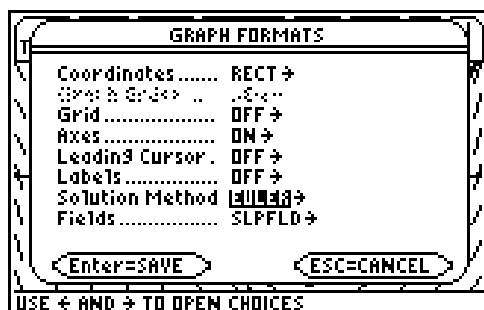


Fig. 11

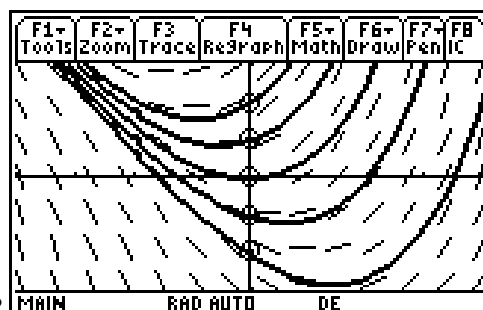


Fig. 12

Look back at figure 1. Would you have guessed that 5 specific solutions look like those shown in figure 12? Adding the Euler solution curves in figure 12, you can almost guess at a symbolic (exact) solution. All solutions appear to approach a line as  $x \rightarrow -\infty$  and grow quickly once the y-axis is passed. Could part of the solution be the line  $y = -x$  and part be  $y = e^x$ ? Could  $y = -x + e^x$  be a solution? Well, no, but sort of...

*Exercise 2:* The fact is that a general solution is  $y = Ce^{x/2} - 2x - 4$ . As usual, for any  $C$  you get a different specific solution. You can easily verify that  $\frac{dy}{dx} = \frac{y}{2} + x$ . Do so.

After you do it by hand, use the home screen **F3 (Calc)** option for derivative, if you want.

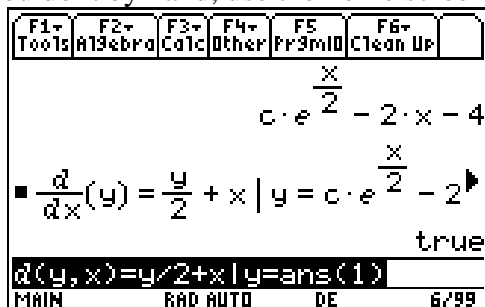


Fig. 12a

*Exercise 3:* For which value of  $C$  does the specific solution go through the origin (the middle solution in figure 12)?

*Exercise 4:* From exercises 2 and 3, you know the equation of a specific solution to the original differential equation. In **DIFF EQUATIONS MODE**, graph the slope field again and save the picture. To do so, on the graph screen, press **F1 2 (Save) ENTER**, make the **Type Picture** (not the default, **GDB**), move the cursor to the **Variable** box, and enter a name, like **sf**. See figure 13. Note the **WINDOW** you're in, then go to **FUNCTION Graph MODE**, make **y1** your function from exercises 3 and 4 and graph it in the same window. The recall the picture. To do so, press **F1 1 (Open) ENTER**, make the **Type Picture**, and select the picture's name in the **Variable** box,. Does the specific solution clearly fit the slope field as in figure 14? (If not, you might have miscalculated  $C$  in Exercise 3.)

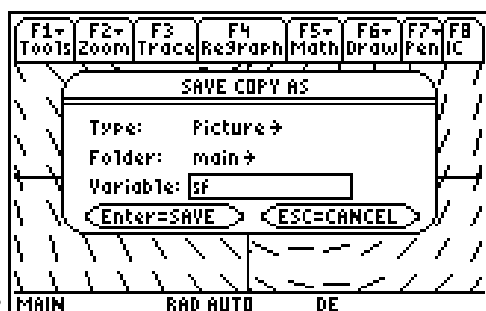


Fig. 13

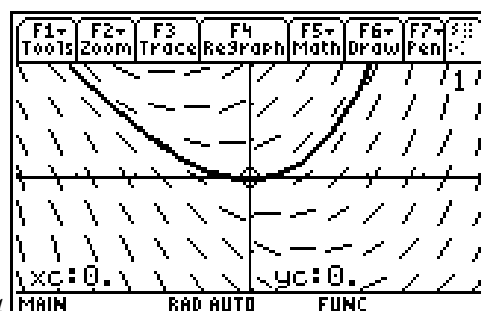


Fig. 14

While **ncurves** (fig. 10 and 12) affords a quick way to get several specific solutions, it takes the process out of your hands. But pressing **F8 (IC)** lets you set initial conditions interactively, one after another. Just enter the coordinates of the initial condition (or move the cursor to it) and press **ENTER**.

*Exercise 5:* Go back to **DIFF EQUATIONS MODE** and draw three specific solution curves.

- Get a line as one of the solutions.
- For which value of  $C$  will you get a line from the general solution?
- Is there another line you could get? Why?
- For the next initial condition, aim for the point  $(1,2)$ .
- Find the  $C$  that will make the solution pass through  $(1,2)$ .
- Finally, make the result go beneath the line you got for the first solution.

Did you get something like the following picture?

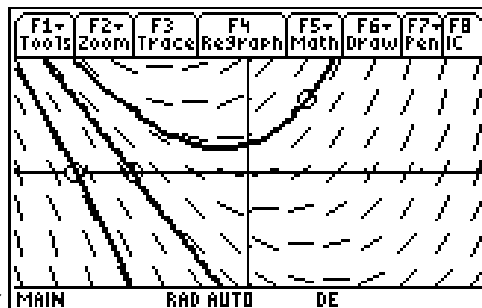


Fig. 15

The following programs implement Euler's method to draw the Euler solution in figure 3 and to make the table of Euler numbers in figure 2. Studying them may help clarify the method. Both programs require the initial condition to be stored as  $(x,y)$  with  $dx$  and  $n$  also input. Before running, the function  $dydx$  must be defined to be the right hand side of the differential equation (as shown in figure 8).

```
EulerPic(x,y,dx,n)
Prgm
  For k,1,n
    x»x0
    y»y0
    y+dydx(x,y)*dx»y
    x+dx»x
    Line x,y,x0,y0
  EndFor
EndPrgm
```

```
EulerTbl(x,y,dx,n)
Prog
  For k,1,n
    x»xv[k]
    y»yv[k]
    x+dx»x
    y+dydx(x,y)*dx»y
  EndFor
```

EndFunc

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Calculus Generic Scope and Sequence Topics: Differential Equations

NCTM Standards: Number and operations, Algebra, Geometry, Problem solving, Reasoning and proof, Connections, Representation