Calculus Investigation

A Mean Slope

TEACHER SOLUTIONS

Part 1: Understanding the Mean Value Theorem

The **Mean Value Theorem** for differentiation states that if $f(x)$ is defined and continuous over the interval $[a, b]$, and differentiable over the interval (a, b), then there is at least one number, c, in the interval (a, b), such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This means that there must be a point, c, where the gradient of the tangent to f is equal to the average gradient.

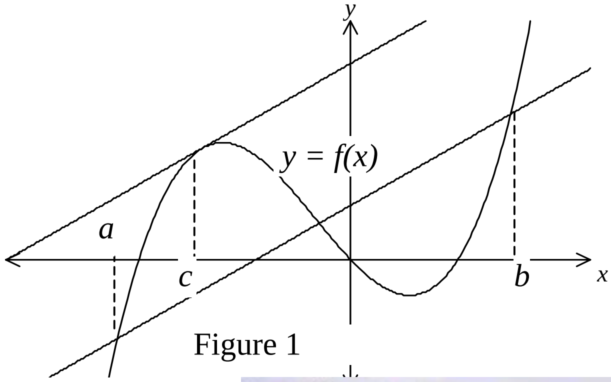

Figure 1

1. Point to point speed cameras that operate on the Hume Freeway are networked and synchronised to measure the average speed of a vehicle over a long distance.

   A truck is detected passing a camera at Seymour at 8:35 am. A camera at Baddaginnie again detects it at 9:16 am, 85 km from Seymour. The speed of the truck past each cameras was 110 km/h.

(a) Calculate the average speed of the truck, in km/h, to the nearest integer.

SOLUTION

(a) t = 41 min = 41/60 h.

$$v = \frac{d}{t} = \frac{85}{41/60} \approx 124.39$$

The average speed was 124 km/h, correct to the nearest integer.

(b) Assuming that the truck did not stop between Seymour and Baddaginnie and that the cameras are accurate, explain, in terms of the Mean Value Theorem, whether the truck driver can be legitimately fined for exceeding the 110 km/h speed limit.

SOLUTION

(b) By the MVT, there is at least one point when the instantaneous speed is equal to the average speed. Therefore, at least once, the truck's speed must have been 124 km/h. Therefore the police can allege that the truck travelled at 124 km/h. Hence the driver can be fined for exceeding 110 km/h.

(c) The driver's lawyer challenged the fine on the grounds that the Mean Value Theorem doesn't apply, because the truck stopped for 5 minutes between Seymour and Baddaginnie. Is this a valid defence? Explain your answer.

SOLUTION

(c) This is not a valid defence, as there was no discontinuity in time. Furthermore, if stopped for 5 minutes, the trip was completed in only 36 minutes of travel time. Hence the average speed, while the truck was in motion, was even higher than 124 km/h.

2. Consider the function f, shown in Figure 1, over the interval $a \le x \le b$.
   (a) Write the coordinates of the endpoints, in terms of a and b.
   (b) Write an expression for the gradient of the line segment joining the endpoints, in terms of a and b.
   (c) Write an expression for the gradient of the tangent to f at the point $(c, f(c))$.

SOLUTION

(a) Endpoints are $(a, f(a))$ and $(b, f(b))$.

(b) $m = \dfrac{f(b) - f(a)}{b - a}$

(c) The gradient of the tangent at $x = c$ is $f'(c)$. If the gradient of the tangent at c is parallel to the line segment passing through the endpoints, then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

3. Consider the function g: $[0, 4] \to R$, where $g(x) = x^3 - 6x^2 + 9x + 2$.

(a) Calculate the average gradient of the function on the interval [0, 4].

SOLUTION

$m = \dfrac{g(4) - g(0)}{4 - 0}$

$m = \dfrac{6 - 2}{4} = 1$

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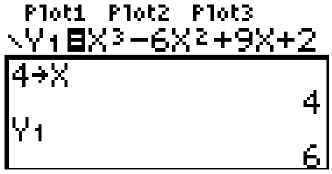

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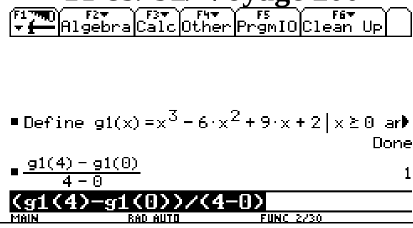


(b) Find the coordinates of **all** points in g where the gradient of the tangent is equal to the average gradient. Approximate the values of the coordinates to two decimal places.

SOLUTION

The gradient of the tangent equals 1

$g'(c) = 3c^2 - 12c + 9 = 1$

$\therefore c = \dfrac{6 \pm 2\sqrt{3}}{3}$

Coordinates are:

$\left(\dfrac{6 \pm 2\sqrt{3}}{3}, g\left(\dfrac{6 \pm 2\sqrt{3}}{3}\right)\right)$

$\therefore \left(\dfrac{6 \pm 2\sqrt{3}}{3}, \dfrac{36 \pm 10\sqrt{3}}{9}\right)$

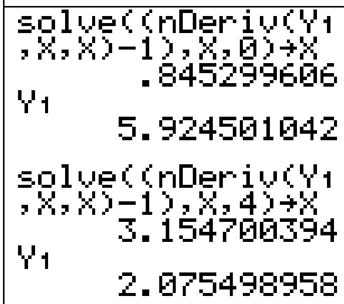
Correct to two decimal places:

$g(0.8453) = 5.0792$

$g(3.1547) = 2.0755$

$\therefore$ (0.85, 5.08) and (3.15, 2.08)

TI-83



‘solve’ in ‘catalog’ menu
solve(*function, variable, guess*)
or use “solver” in the MATH menu.

Keystrokes:

[2nd][0](Catalog). Select ‘solve’ [(][(][MATH][8]:nDeriv([VARS][▸] :Y-Vars [1][,][X,T,Θ,*n*][,] [X,T,Θ,*n*][−][1][)][,][X,T,Θ,*n*] [,][1][.][2][)][ENTER]

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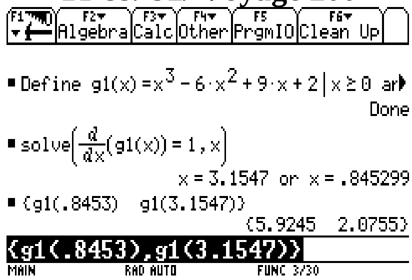
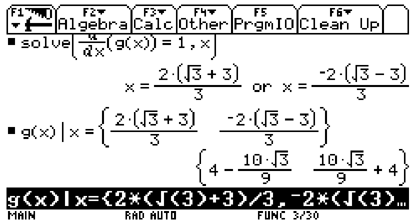
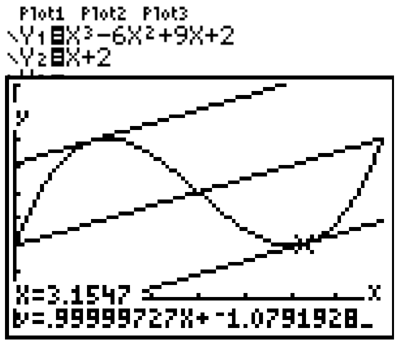


Exact coordinates



(c) Sketch and label a graph of g, showing the line segment of average gradient and the tangents. Label the coordinate(s) of the turning point(s) and the coordinate(s) of the point(s) of intersections of g and the tangents, to two decimal places

TI-83



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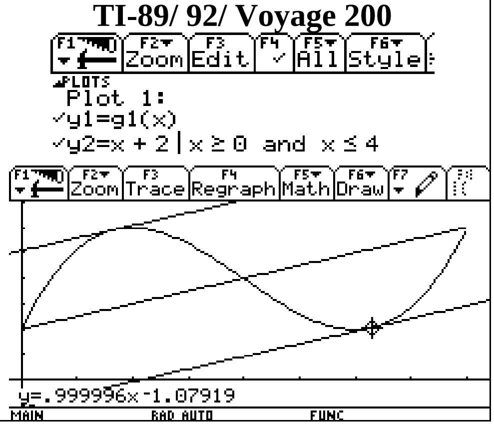


Part 2: Applying the Mean Value Theorem for Differentiation

4. An amusement park ride has a platform that moves up and down, on a tower, with increasing amplitude. For part of the ride, the height of the platform, h metres above the ground, at time t seconds, is modelled by the function
h: [0, 11] $\rightarrow R$, $h(t) = pt\cos(5\sqrt{(t+1)}) + q$,
where the parameters p and q have positive real values.

(a) The initial height of the platform is 10 m, and at t = 11 seconds the height is 10.46 m. Find the value of p and q, correct to the nearest integer. Hence write the rule for $h(t)$.

SOLUTION

$h(t) = pt\cos(5\sqrt{(t+1)} + q$

$h(0) = 10$

$\therefore\ p \times 0 \times \cos(5\sqrt{(0+1)} + q = 10$

$\therefore\ q = 10$

$h(11) = 10.4$

$\therefore\ 11p\cos(5\sqrt{(11+1)} + 10 = 10.4$

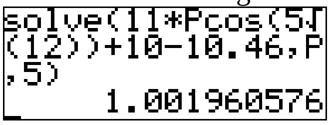
$\therefore\ p = \dfrac{10.46-10}{11\cos(5\sqrt{12})} \approx 1.002$

Correct to the nearest integer,

$h(t) = t\cos(5\sqrt{(t+1)}) + 10$

TI-83

'solve' from 'catalog' menu



Syntax is:
solve(*function, variable, guess*)
Alternatively, use 'solver' from 'math' menu.

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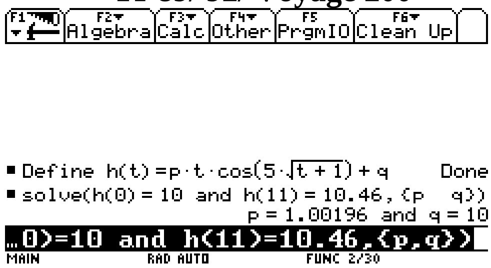

(b) Sketch the graph of h, over the specified domain, labelling the coordinates of the endpoints, correct to two decimal places. (There is no need to work out the coordinates of the turning points).

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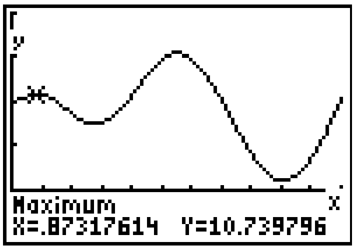

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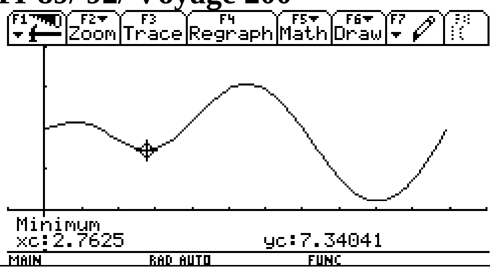


(c) Comment on the key features of the graph.

SOLUTION
The graph is continuous and smooth on [0,11], and therefore differentiable on (0,11). It has two local maxima and two local minima.

(d) At what times, in seconds, correct to two decimal place, is the velocity of the platform equal to zero?

SOLUTION

Velocity is zero $\therefore$ $h'(t) = 0$

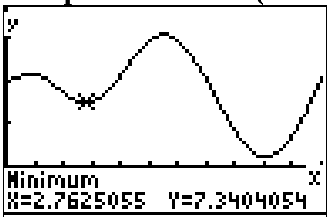
Graphically, find coordinates of the stationary points.

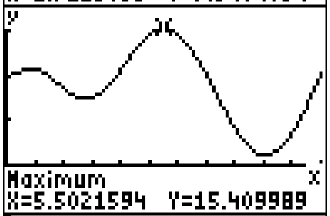
Numerically, solve $h'(t) = 0$, using appropriate bounds.

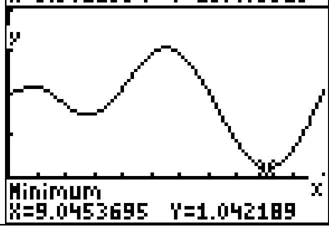
Correct to 2 decimal places, $h'(t) = 0$ for

***t* = 0.87s, *t* = 2.76 s, *t* = 5.50 s, *t* = 9.05 s**

Graphical solution (All TIs)







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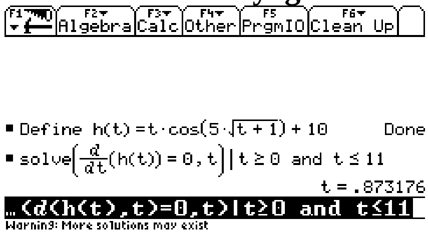
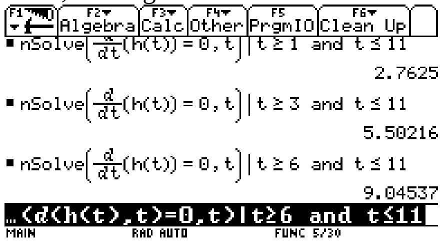


Only one solution found algebraically

To find the other solutions, use numerical solve, shifting the bounds each time.



The graphical method is identical to that of the TI-83.

(e) What is the height of the platform, in metres, correct to two decimal place, at the times when the velocity of the platform is zero?

SOLUTION

The heights can be found graphically, or numerically, by substituting the values of t into $h(t)$.

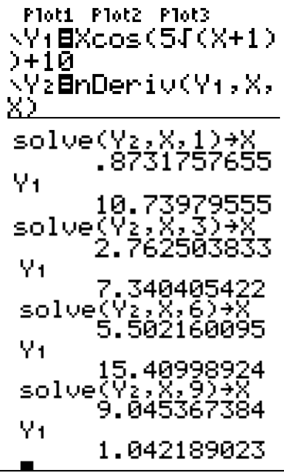
$h(0.87) = 10.74$ m

$h(2.76) = 7.34$ m

$h(5.50) = 15.41$m

$h(9.05) = 1.04$ m

TI-83 Numerical solution



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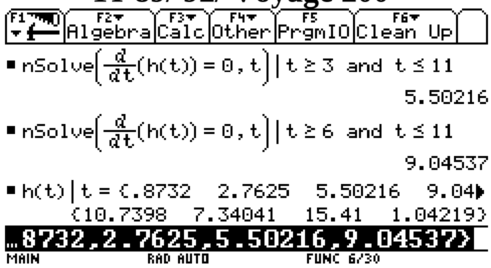
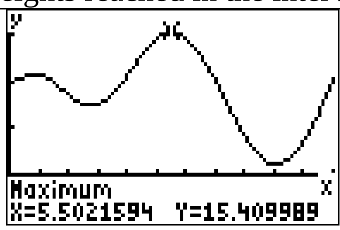


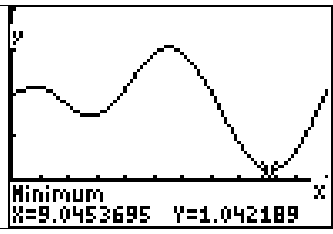
(f) What are the maximum and minimum heights reached in the interval $0 \le t \le 11$?

SOLUTION

The maximum height reached is 15.41 m.

The minimum height reached is 1.04 m.





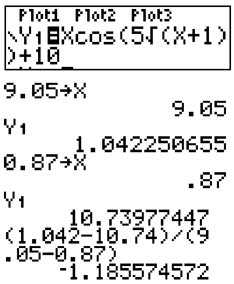
(g) Calculate the average velocity of the platform, correct to two decimal places, on $t \in [0.87, 9.05]$.

SOLUTION

$$v_{av} = \frac{h(9.05) - h(0.87)}{9.05 - 0.87} = -1.19$$

The average velocity is – 1.19 m/s.

TI-83



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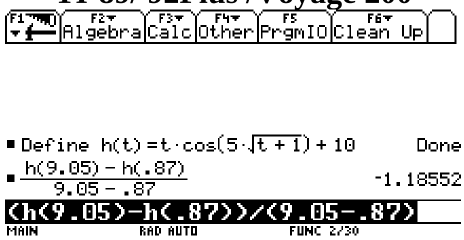


(h) Find the equation of the line joining the endpoints of the interval [0.87, 9.05]. Give your answers correct to two decimal places.

SOLUTION

The equation of the line segment:

$m = v_{av} = -1.19$

and passes through (0.87, 10.74)

$$y - y_1 = m(x - x_1)$$

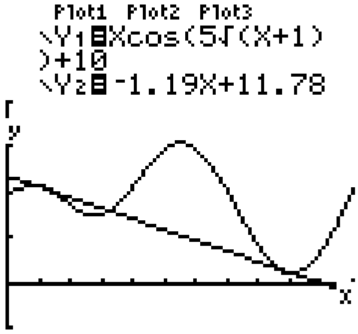
$$y - 10.74 = -1.19(x - 0.87)$$

$$\therefore \quad y = -1.19x + 11.78$$

Equation of the line is

$h(t) = -1.19\, t + 11.78$

TI-83



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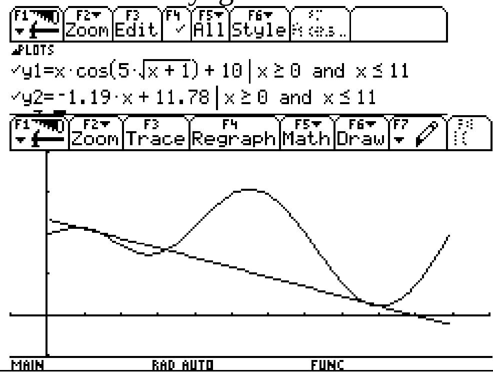


(i) Find the times, correct to two decimal places, at which the instantaneous velocity of the platform is equal to the average velocity over the interval [0.87, 9.05].

SOLUTION

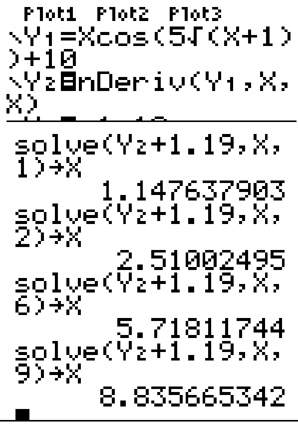
The instantaneous velocity of the platform is equal to the average velocity when

$h'(t) = -1.19$

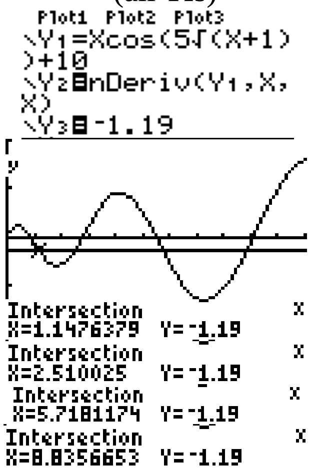
Graphically, or solving numerically,

t = 1.15 s, 2.51 s, 5.72 s, 8.84 s.

TI-83 Numerical Solution



Graphical solution (all TIs)



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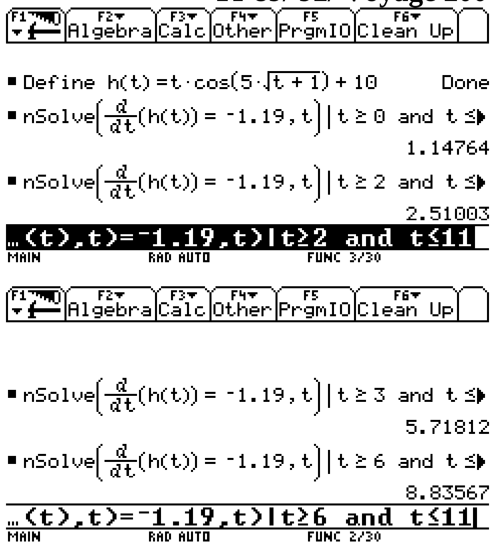
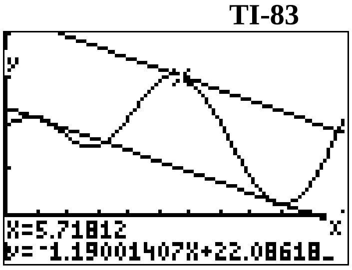


(j) Sketch and label a graph of h, showing the line segment joining the endpoints of the interval [0.87, 9.05] and one of the tangents to h that represent the instantaneous velocity of the platform being equal to the average velocity over the interval [0.87, 9.05].

TI-83



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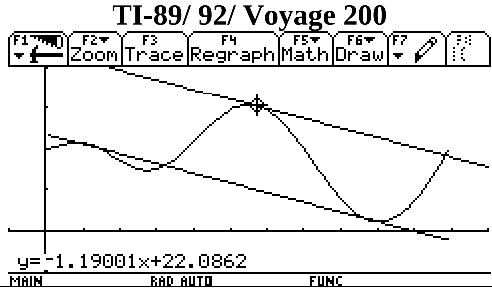


Part 3: Mean Value Theorem as a generalisation of Rolle's Theorem

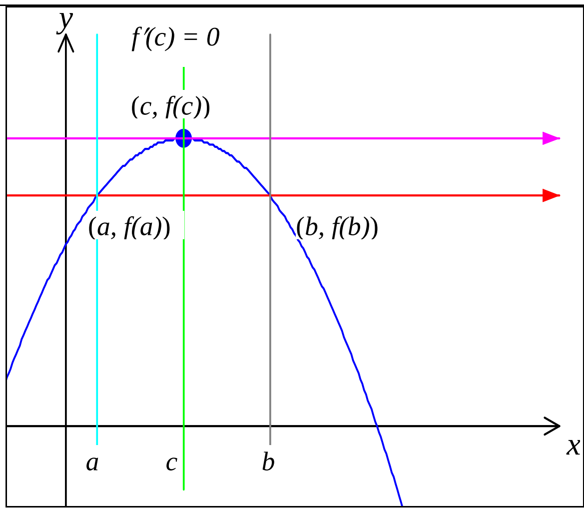
The Mean Value Theorem is a generalisation of another theorem, called Rolle's Theorem

Rolle's Theorem: let f be defined and continuous over the interval $[a, b]$, and differentiable over the interval (a, b). If $f(a) = f(b)$, then there is at least one point, c, in the interval (a, b) for which $f'(c) = 0$.

5. Explain the geometric meanings of $f(a) = f(b)$ and $f'(c) = 0$, where $c \in (a, b)$. Hence illustrate Rolle's Theorem geometrically.

SOLUTION

$f(a) = f(b)$ means that, at $x = a$ and at $x = b$, f has the same y-value. $f'(c) = 0$ represents a stationary point at $x = c$.
Hence Rolle's theorem says that if f is a 'smooth' function and the y-values at $x = a$ and at $x = b$ are equal, then there must be a stationary point at $x = c$, between a and b.



6. To prove the mean value theorem, consider the function

$$g(x) = \left((x-a)\frac{f(b)-f(a)}{b-a} + f(a) \right) - f(x), \quad x \in [a,b]$$

(a) (i) If $f(x) = x^2$, find the equation of $g(x)$ for $a = 1$ and $b = 3$.

(ii) On the same set of axes, draw the graphs of $f(x)$, $g(x)$ and the line segment passing through the points $(a, f(a))$ and $(b, f(b))$.

SOLUTION

$$g(x) = \left((x-a)\frac{f(b)-f(a)}{b-a} + f(a) \right) - f(x)$$

$$\therefore g(x) = \left((x-1)\frac{3^2-1^2}{3-1} + 1^2 \right) - x^2$$

$$\therefore g(x) = (4(x-1)+1) - x^2$$

$$\therefore g(x) = -x^2 + 4x - 3$$

The line passing through (1, 1) and (3, 9)

$$\therefore \quad m = \frac{9-1}{3-1} = 4$$

$\therefore y - 1 = 4(x - 1)$

$\therefore y = 4x - 3$

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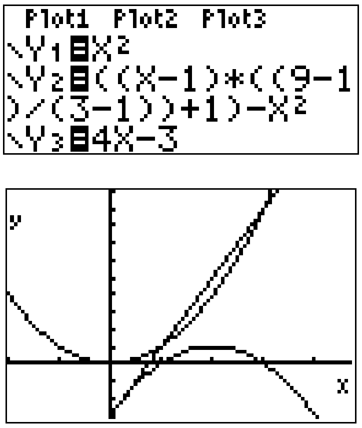

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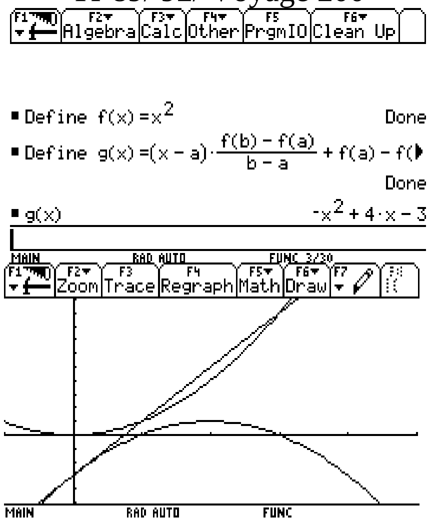

(b) From the situation illustrated in part (a) above, give a geometric interpretation to the function g, defined on $[a, b]$.

SOLUTION

Note that $y = (x-a)\dfrac{f(b)-f(a)}{b-a} + f(a)$ is just the equation of the straight line passing through $(a, f(a))$ and $(b, f(b))$. That is, $y - y_1 = m(x - x_1)$, where $x_1 = a$, $y_1 = f(a)$ and $m = \dfrac{f(b)-f(a)}{b-a}$.

Hence, for any values a and b, let $h(x)$ be the equation of the line passing through $(a, f(a))$ and $(b, f(b))$.
In all cases, $g(x) = h(x) - f(x)$. Hence, geometrically, $g(x)$ can be obtained by addition of ordinates: $h(x) + (-f(x))$.
$g(x)$ is a transformation of $f(x)$ such that $f(x)$ is reflected in the x-axis, and translated such that the x-intercepts are $(a, 0)$ and $(b, 0)$.

(c) If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), explain why there must be a point c in (a, b) such that $g'(c) = 0$.

SOLUTION
The x-intercepts of $g(x)$ are always $(a, 0)$ and $(b, 0)$. Since $f(x)$ is a "smooth" function, by Rolle's Theorem, there must be a value c on the interval (a, b), such that $g'(c) = 0$.

(d) If $g(x) = \left((x-a)\dfrac{f(b)-f(a)}{b-a} + f(a)\right) - f(x),\ \ x \in [a,b]$, find $g'(x)$. Hence find $g'(c)$.

SOLUTION

$$g(x) = \left((x-a)\frac{f(b)-f(a)}{b-a} + f(a)\right) - f(x)$$

$$\therefore g'(x) = \left(\frac{f(b)-f(a)}{b-a} + 0\right) - f'(x)$$

$$\therefore g'(c) = \left(\frac{f(b)-f(a)}{b-a}\right) - f'(c)$$

TI-83

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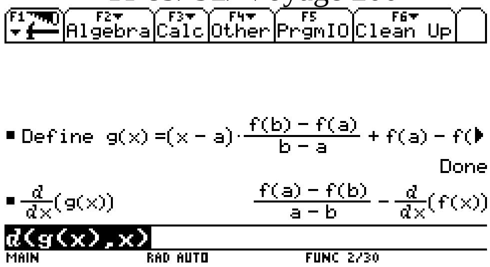


(e) Given that $g'(c) = 0$, rearrange the expression in part (d) above, to make $f'(c)$ the subject of the equation.

SOLUTION
$g'(c) = 0$, from Rolle's Theorem

$$\therefore 0 = \left(\frac{f(b)-f(a)}{b-a}\right) - f'(c)$$

$$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}$$

(f) Explain the significance of the result obtained in part (e) above.

SOLUTION
The result is an expression of the Mean Value Theorem (MVT). Thus, MVT can be proved from Rolle's Theorem. Rolle's theorem is a special case of the MVT, where the average gradient of the function, on $[a, b]$, is zero.

Part 4: Mean Value Theorem for Integration and Average Value of a function

Let $y = f(x)$ be a function which is continuous on the closed interval $[a, b]$. Then there exists at least one value, c, in the interval $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a) \quad \ldots \text{ equation 1}$$

$f(c)$ gives the **average value** of f from $x = a$ to $x = b$. Rearranging equation 1:

$$f(c) = \frac{1}{b-a}\int_a^b (f(x))dx \quad \ldots \text{ equation 2}$$

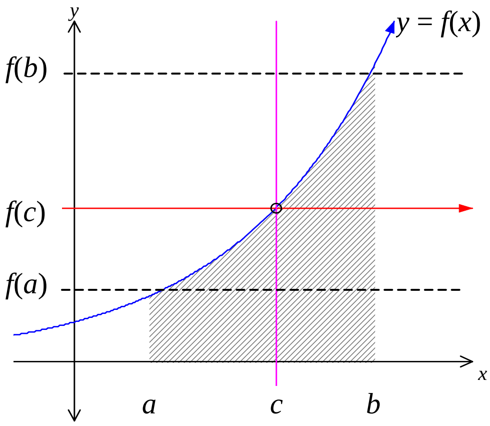


Figure 2.

7. Consider the relationship between Figure 2 and equation 2.
   (a) Explain the geometric meaning of the following, with reference to Figure 2:

   (i) $\displaystyle\int_a^b (f(x))dx$ (ii) $b-a$

   (b) Hence explain why $\dfrac{1}{b-a}\displaystyle\int_a^b (f(x))dx$ gives the average value of f on the interval $[a, b]$.

 PD scholarship winner 2004 Frankston High School, Victoria

SOLUTION

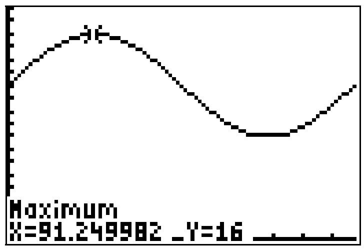
(a) (i) $\int_a^b (f(x))dx$ gives the area bounded by the curve, the x-axis, $x = a$ and $x = b$. That is, the area of the shaded region Figure 2.
(ii) $(b - a)$ is the width of the shaded region in Figure 2.
(b) The area of the shaded region is given by the width multiplied by the average value (average "height").
Therefore, the area divided by the width, $\frac{1}{b-a}\int_a^b (f(x))dx$, gives the average value of g on [a, b]. (This is analogous to rearranging the formula for area of a trapezium to find the average length of the parallel sides).

8. At a certain latitude, the length of time, x hours, from sunrise to sunset, t days after a Spring equinox, is modelled by the function $x(t) = 4\sin\left(\frac{2\pi}{365}t\right) + 12$, $0 \le t \le 365$.

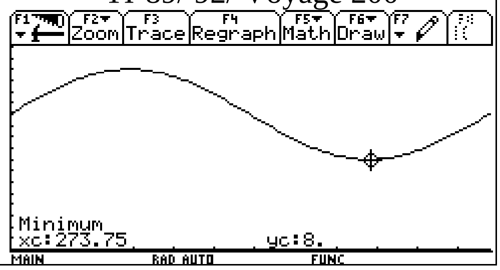
(a) What are the maximum and minimum hours of daylight, according to this model?

SOLUTION
Maximum = 12 + 4 = 16 hours
Minimum = 12 – 4 = 8 hours

TI-83



TI-89/ 92/ Voyage 200



(b) Use the *average value of a function* formula to calculate the average number of daylight hours per day during the 182.5 days following the Spring Equinox (i.e. over $0 \le t \le 182.5$). Give the **exact** value, in hours, and an approximation to two decimal places.

SOLUTION
The integral can be evaluated 'by hand' or with the use of technology.

$$\frac{1}{182.5}\int_0^{182.5}\left(4\sin\left(\frac{2\pi t}{356}\right)+12\right)dt$$

$$= \frac{1}{182.5}\left[-\frac{730}{\pi}\cos\left(\frac{2\pi t}{356}\right)+12t\right]_0^{182.5}$$

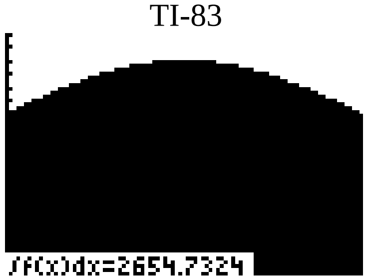
$$= \frac{8}{\pi}+12$$

$$= \frac{4(3\pi+2)}{\pi}$$

$$\approx 14.55$$

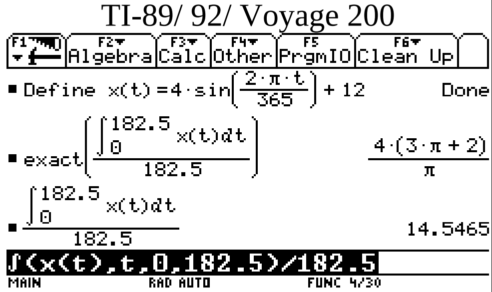
The average length of daylight is 14.55 hours (2 decimal places).

TI-83

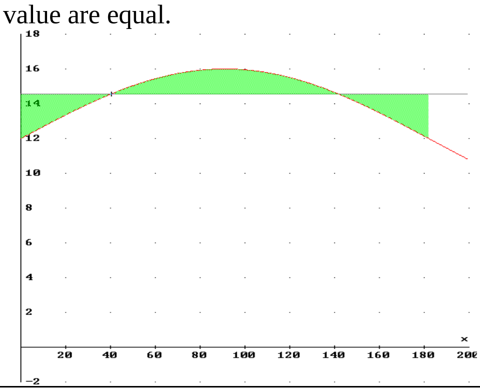


2654.7325/182.5
14.54647945

TI-89/ 92/ Voyage 200



The shaded areas above and below the mean value are equal.



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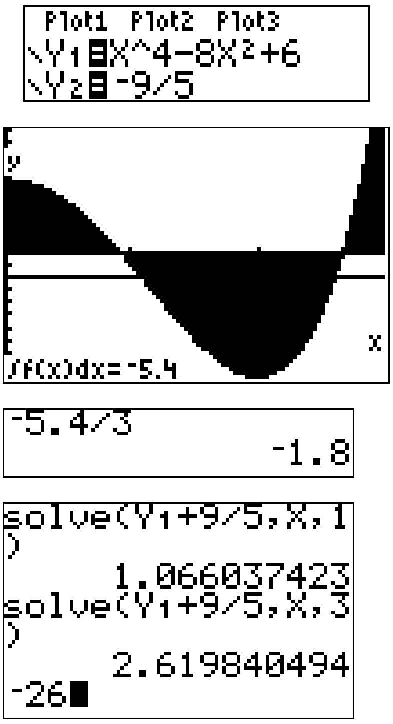
9. Consider the function $g(x) = x^4 - 8x^2 + 6,\ 0 \le x \le 3$.
   (a) Find $g(c)$, the average value of g on [0, 3]
   (b) Find all values of c, correct to 2 decimal places.
   (c) On a graph of g, for $x \in [0, 3]$, shade the regions representing $\int_0^3 (g(x))dx$. On the same axes, draw the graph of $y = g(c)$. Label the coordinates of the endpoints and the coordinates of all values of c.

<table>
<tr><th>SOLUTION</th><th>TI-83</th><th>TI-89/ 92/ Voyage 200</th></tr>
<tr>
<td>(a) The integral can be evaluated ‘by hand’ or with the use of technology.
$\frac{1}{b-a}\int_a^b (f(x))dx = \frac{1}{3-0}\int_0^3 (x^4 - 8x^2 + 6)dx$
$g(c) \quad = \frac{1}{3}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 6x\right]_0^3$
$g(c) \quad = \frac{1}{3}\left[\frac{3^5}{5} - \frac{8\times 3^3}{3} + 6\times 3\right] - \left[\frac{0}{5} - \frac{0}{3} + 0\right]$
$g(c) = \frac{-9}{5}$
The average value is $\frac{-9}{5}$ *or* -1.8
(b) To find all values of c, solve $g(x) = -1.8$. Using technology,
$x^4 - 8x^2 + 6 = -1.8$
$\therefore \quad x = 1.07 \ \ or \ \ x = 2.62$
The values of c are 1.07 or 2.62</td>
<td>
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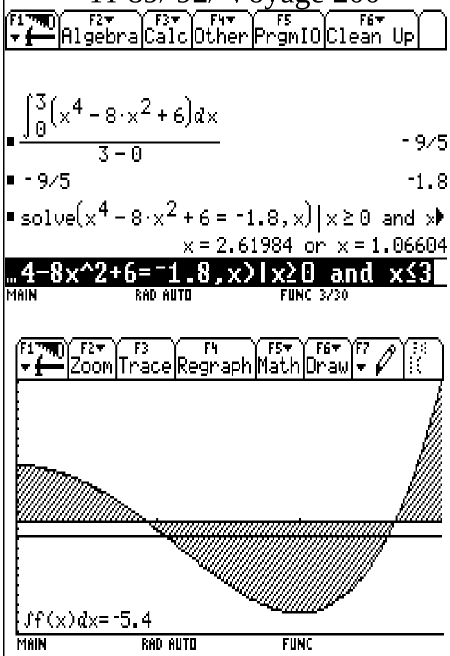
<td>
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Calculus Investigation: Student Booklet

A Mean Slope

NAME:

Part 1: Understanding the Mean Value Theorem for differentiation

The **Mean Value Theorem** for differentiation states that if $f(x)$ is defined and continuous over the interval $[a, b]$, and differentiable over the interval (a, b), then there is at least one number, c, in the interval (a, b), such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This means that there must be a point, c, where the gradient of the tangent to f is equal to the average gradient.

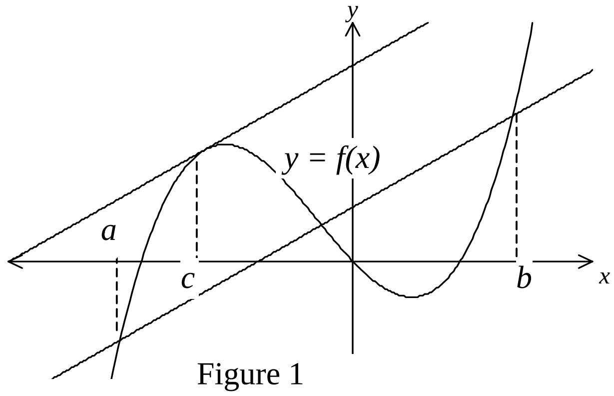


Figure 1

1. Point to point speed cameras that operate on the Hume Freeway are networked and synchronised to measure the average speed of a vehicle over a long distance.

   A truck is detected passing a camera at Seymour at 8:35 am. A camera at Baddaginnie again detects it at 9:16 am, 85 km from Seymour. The speed of the truck past each cameras was 110 km/h.

(a) Calculate the average speed of the truck, in km/h, to the nearest integer.

(b) Assuming that the truck did not stop between Seymour and Baddaginnie, and that the cameras are accurate, explain, in terms of the Mean Value Theorem, whether the truck driver can be legitimately fined for exceeding the 110 km/h speed limit.

(c) The driver's lawyer challenged the fine on the grounds that the Mean Value Theorem doesn't apply, because the truck stopped for 5 minutes between Seymour and Baddaginnie, creating a discontinuity. Is this a valid defence? Explain your answer.

2. Consider the function f, shown in Figure 1, over the interval $a \le x \le b$.

(a) Write the coordinates of the endpoints, in terms of a and b.

(b) Write an expression for the gradient of the line segment joining the endpoints, in terms of a and b.

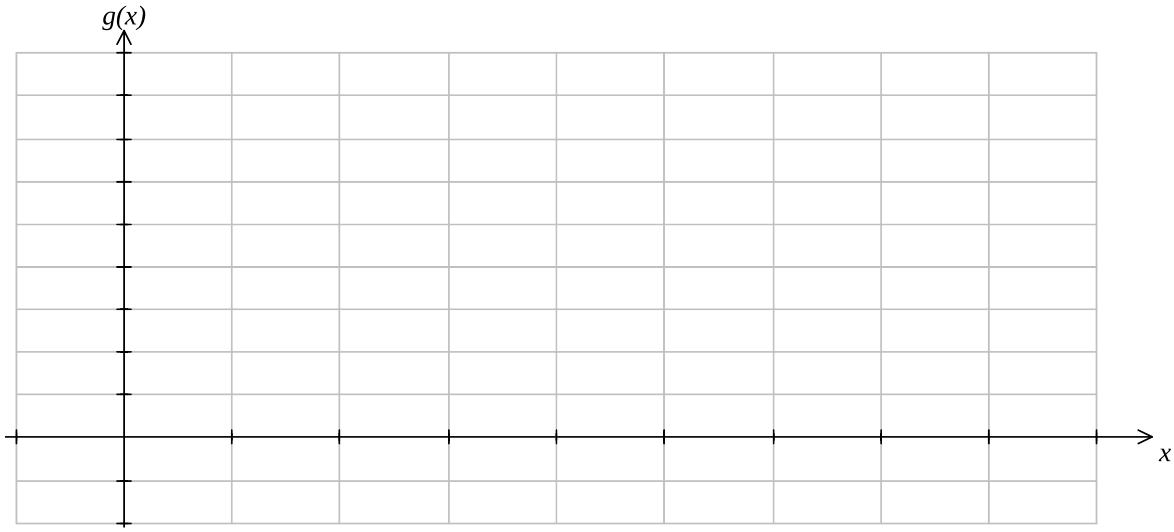
(c) Write an expression for the gradient of the tangent to f at the point (c, $f(c)$), in terms of a and b.

3. Consider the function g: $[0, 4] \to R$, where $g(x) = x^3 - 6x^2 + 9x + 2$.

(a) Calculate the average gradient of the function on the interval [0, 4].

(b) Find the coordinates of all points in g where the gradient of the tangent is equal to the average gradient. Approximate values of the coordinates to two decimal places.

(c) Sketch and label a graph of g, showing the line segment of average gradient and the tangents. Label the coordinate(s) of the turning point(s) and the coordinate(s) of the point(s) of intersections of g and the tangents, to two decimal places.



Part 2: Applying the Mean Value Theorem for Differentiation

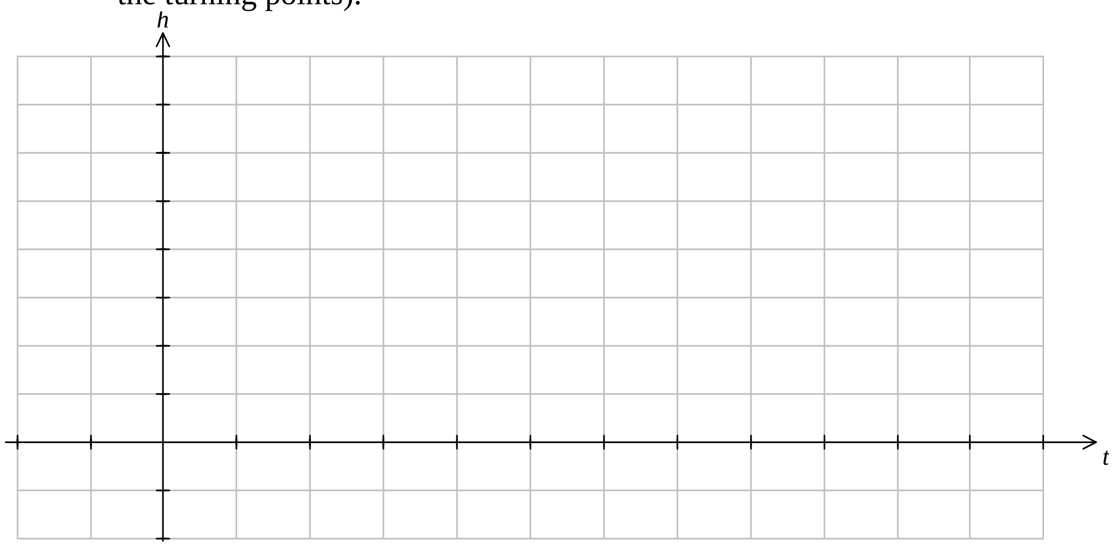
4. An amusement park ride has a platform that moves up and down, on a tower, with increasing amplitude. For part of the ride, the height of the platform, h metres above the ground, at time t seconds, is modelled by the function

   $h: [0, 11] \rightarrow R,\ h(t) = pt\cos(5\sqrt{(t+1)}) + q$,

   where the parameters p and q have positive real values.

(a) The initial height of the platform is 10 m, and at t = 11 seconds the height is 10.46 m. Use these data to find the value of p and q, correct to the nearest integer. Hence write the rule for $h(t)$.

(b) Sketch the graph of h, over the specified domain, labelling the coordinates of the endpoints, correct to two decimal places. (There is no need to work out the coordinates of the turning points).



(c) Comment on the key features of the graph.

(d) At what times, in seconds, correct to two decimal place, is the velocity of the platform equal to zero?

(e) What is the height of the platform, in metres, correct to two decimal place, at the times when the velocity of the platform is zero?

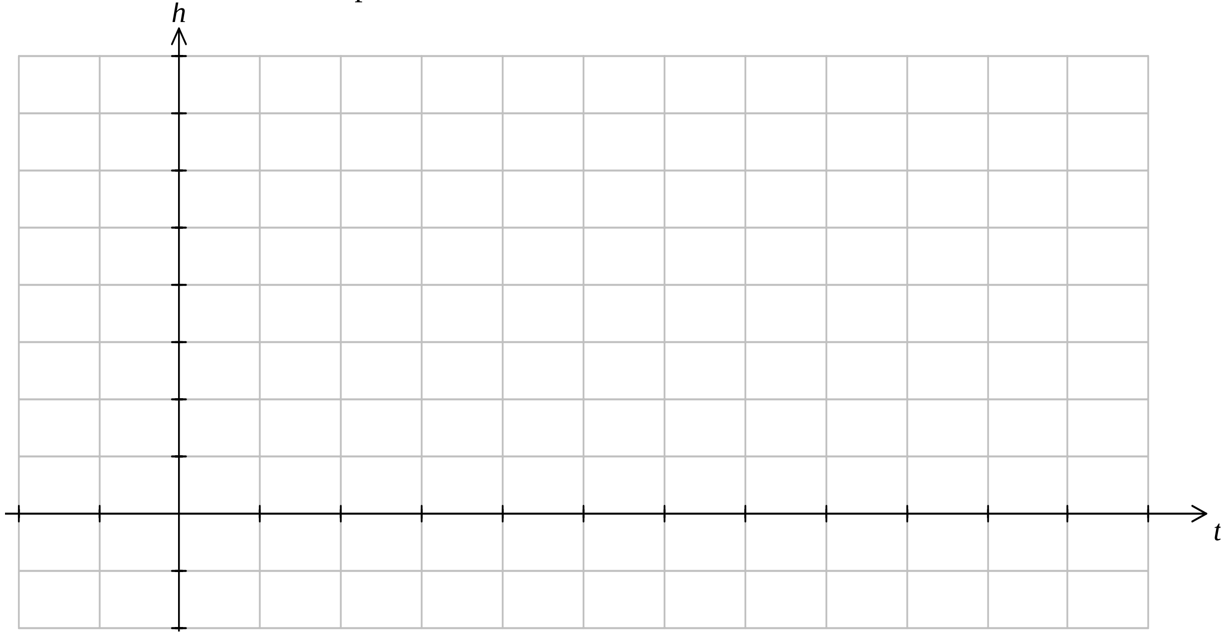
(f) What are the maximum and minimum heights reached in the interval $0 \leq t \leq 11$?

(g) Calculate the average velocity of the platform, correct to two decimal places, on $t \in [0.87, 9.05]$.

(h) Find the equation of the line joining the endpoints of the interval [0.87, 9.05]. Give your answers correct to two decimal places.

(i) Find the times, correct to two decimal places, at which the instantaneous velocity of the platform is equal to the average velocity over the interval [0.87, 9.05].

(j) Sketch and label a graph of h, showing the line segment joining the endpoints of the interval [0.87, 9.05] and the tangents to h that represent the instantaneous velocity of the platform is equal to the average velocity over the interval [0.87, 9.05]. Include the coordinates of the points where the touch the curve.



Part 3: Mean Value Theorem as a generalisation of Rolle's Theorem

The Mean Value Theorem is a generalisation of another theorem, called Rolle's Theorem

Rolle's Theorem: let f be defined and continuous over the interval $[a, b]$, and differentiable over the interval (a, b). If $f(a) = f(b)$, then there is at least one point, c, in the interval (a, b) for which $f'(c) = 0$.

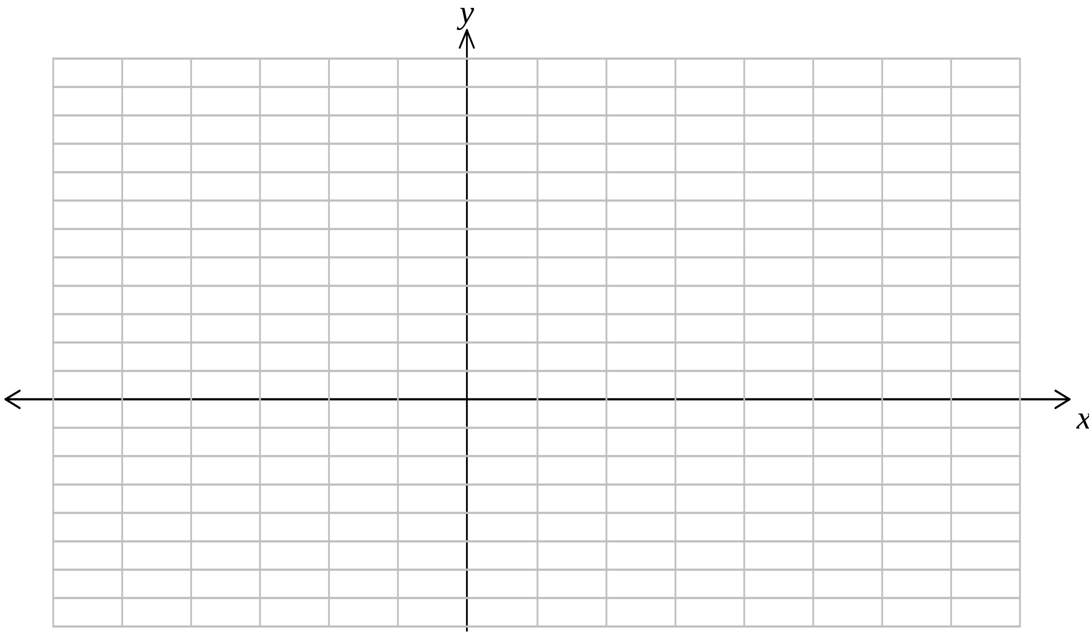
5. Explain the geometric meanings of $f(a) = f(b)$ and $f'(c) = 0$, where $c \in (a, b)$. Hence illustrate Rolle's Theorem geometrically.

6. To prove the mean value theorem, consider the function

$$g(x) = \left((x-a)\frac{f(b)-f(a)}{b-a} + f(a) \right) - f(x), \quad x \in [a,b]$$

(a) (i) If $f(x) = x^2$, find the equation of $g(x)$ for $a = 1$ and $b = 3$.

(ii) On the same set of axes, draw the graphs of $f(x)$, $g(x)$ and the line segment passing through the points $(a, f(a))$ and $(b, f(b))$.



(b) From the situation illustrated in part (a) above, give a geometric interpretation to the function g, defined on $[a, b]$.

(c) If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), explain why there must be a point c in (a, b) such that $g'(c) = 0$.

(d) If $g(x) = \left((x-a)\dfrac{f(b)-f(a)}{b-a} + f(a)\right) - f(x),\ \ x \in [a,b]$, find $g'(x)$, find $g'(c)$.

(e) Given that $g'(c) = 0$, rearrange the expression in part (d) above, to make $f\,'(c)$ the subject of the equation.

(f) Explain the significance of the result obtained in part (e) above.

Part 4: Mean Value Theorem for Integration and Average Value of a function

Let $y = f(x)$ be a function which is continuous on the closed interval **[**a, b**]**. Then there exists at least one value, c, in the interval **[**a, b**]** such that

$$\int_a^b f(x)dx = f(c)(b-a) \quad \text{… equation 1}$$

$f(c)$ gives the **average value** of f from $x = a$ to $x = b$. Rearranging equation 1:

$$f(c) = \frac{1}{b-a}\int_a^b (f(x))dx \quad \text{… equation 2}$$

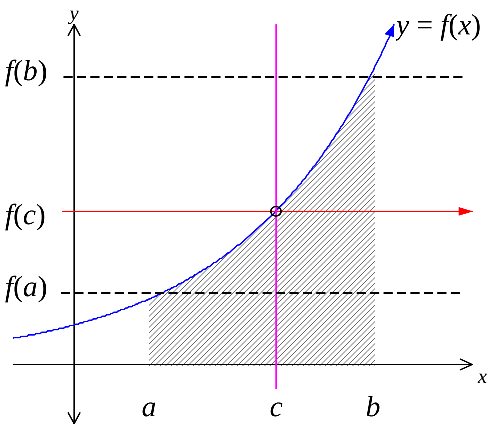


Figure 2.

7. Consider the relationship between Figure 2 and equation 2.
   (a) Explain the geometric meaning of the following, with reference to Figure 2:

   (i) $\int_a^b (f(x))dx$ (ii) $b-a$

(b) Hence explain why $\frac{1}{b-a}\int_a^b (f(x))dx$ gives the average value of f on the interval [a, b].

8. At a certain latitude, the length of time, x hours, from sunrise to sunset, t days after a Spring equinox, is modelled by the function $x(t) = 4\sin\left(\frac{2\pi}{365}t\right)+12$, $0 \le t \le 365$.

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