Eye Spy

Teacher Notes and Answers

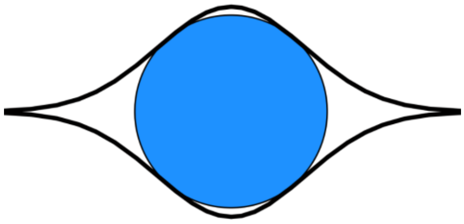
7 8 9 10 **11** 12

TI-Nspire™ Investigation Student 50 min

Problem to Solve

This problem is called "Eye Spy" because the combination of graphs looks a little bit like an eye. The top outline of the eye is defined by the function:

$$f(x) = e^{-x^2}$$



The iris of the eye is defined by the relation:

$$x^2 + y^2 = r^2 \quad \text{where } r \text{ is the radius of the iris.}$$

The aim of this investigation is to determine the maximum area of the iris where it is contained within the outline of the eye.

Question: 1.

Determine a function for the lower outline of the eye.

Answer: $h(x) = -f(x)$ where $h(x)$ represents the lower outline of the eye.

Question: 2.

Let P represent the point where the outer edge of the iris connects with the outline of the eye.

a) Write an equation relating $f(x)$ and the equation to the iris at point P.

Answer: The simplest solution is for students to simply substitute their expression $f(x)$ into the circle relation: $x^2 + y^2 = r^2$ and leave it! Students however often just 'apply' the solve command. In this particular example the Solve command makes the result appear more complicated. In either case students should recognise the problem is symmetrical vertically and horizontally and that the radius must be positive.

$$x^2 + (f(x))^2 = r^2$$

$$x^2 + \left(e^{-x^2}\right)^2 = r^2$$

$$x^2 + e^{-2x^2} = r^2$$

This is the easier option and involves simple substitution (no calculator required).

$$\sqrt{r^2 - x^2} = e^{-x^2}$$

$$r = e^{-x^2}\sqrt{x^2 e^{2x^2} + 1}$$

If students use the solve command the result can appear more complicated.

b) Write an equation for the area of the iris in terms of x.

Answer: The simplest equation: $a(x) = \pi\left(x^2 + e^{-2x^2}\right)$

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c) Use calculus to determine the value of x for which the area of the iris is a maximum given the iris fits completely inside the outline of the eye.

Answer: Note that CAS is not required, but may be used.

Three solutions exist that satisfy the condition of the derivative equal to zero. One pair represents the maximum radius and reinforces the symmetry of the problem. The third solution (x = 0) has the iris so that it just touches the outline of the eye, however the iris goes beyond the eye outline.

Students should comment on which solution(s) are accepted and why.

$$\frac{d(a(x))}{dx} = \pi\left(2x - 4xe^{-2x^2}\right)$$

$$0 = 2\pi x\left(1 - 2e^{-2x^2}\right)$$

$$x = 0 \text{ or } x = \mp\sqrt{\frac{\ln(2)}{2}}$$

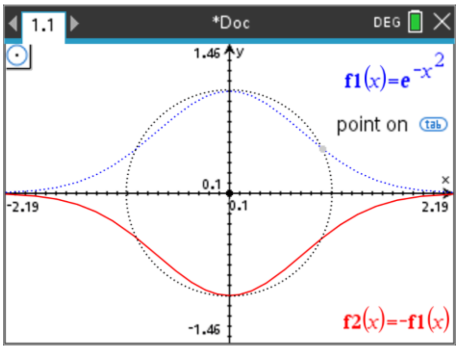
$$Area = \frac{\pi\left(\ln(2)+1\right)}{2}$$

Teacher Notes: It is very easy to create a dynamic representation of this problem that can be used to help students verify that their equation is correct.

Graph the function $y = e^{-x^2}$ and its reflection.

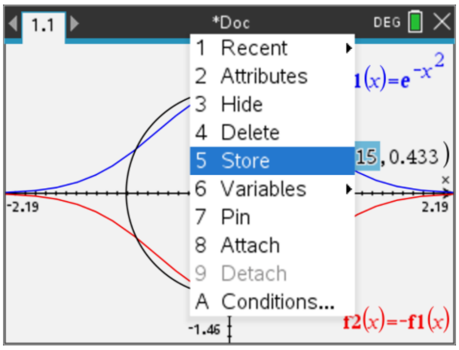
Use the geometry tool to draw a circle centred at the origin and a point on the curve.

Note that this construction does not satisfy the necessary conditions that the iris fit 'inside' the upper and lower bounds of the eye.



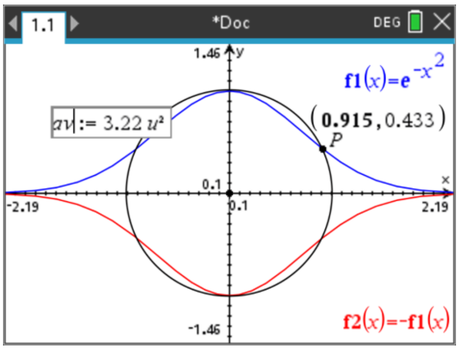
Label the point used to create the circle as P and display the coordinates.

Use the contextual menu [Handheld: Ctrl + Menu or PC: Right Click] on the x coordinate and store this in **xv**.



Use the Geometry tool to measure the area of the circle and store the area in a variable **av**.

Notice that both the x coordinate and the area measurements become bold. Hover the mouse over either value to see how the variable is linked.



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Insert a spreadsheet application and name the first two columns as:

xp & yp

In the formula bar use the menus to capture data automatically.

Menu > Data > Capture > Auto

In column xp, capture the data associated with the x coordinate (xv).

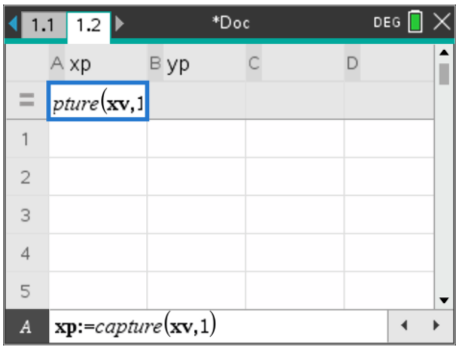
In column yp, capture the data associated with the area (av).

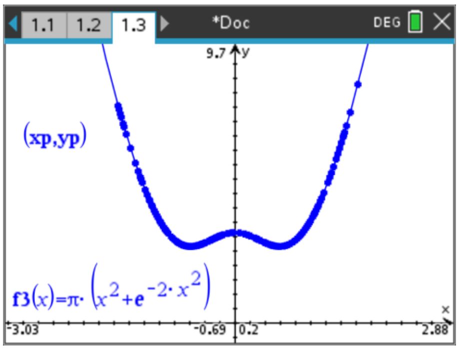
Return to the Graph application and move point P along the curve; data will automatically be recorded in the spreadsheet.

Insert a graph application and produce a scatterplot for xp, yp.

Use student equations to check to ensure they model the area of the iris.

The dynamic visual representation should be aligned to the shape of the graph. The graph can be used to help students understand the solutions to the derivative (next question).





Question: 3.

a) Write an equation relating $f'(x)$ and the gradient of the iris at point P.

Answer: Students could use calculus to work out the gradient of the circle at any point, however a little geometry knowledge leads to a gradient of:

$$\frac{dy}{dx} = -\frac{x}{y}$$

The gradient of the top of the eye:

$$f'(x) = -2xe^{-x^2}$$

The gradient of the circle and the curve will be the same where they touch:

$$-\frac{x}{y} = -2xe^{-x^2}$$

$$x = 2xye^{-x^2}$$

$$x = 2xe^{-2x^2}$$

b) Determine the value of x and corresponding maximum area of the iris.

Answer: Students may miss $x = 0$ through cancellation. Otherwise, as before:

$$0 = x\left(2e^{-2x^2} - 1\right)$$

$$x = 0 \text{ or } x = \mp\sqrt{\frac{\ln(2)}{2}}$$

where: $x = \mp\sqrt{\frac{\ln(2)}{2}}$ produces maximum area: $\frac{(\ln(2)+1)\cdot\pi}{2}$

Teacher Notes:

If students rely on a sketch graph only, it would be easy to imagine the circle just touching the maximum of the function: $f(x)=e^{-x^2}$. The technology allows students to see an accurate representation of the situation and estimate where the maximum might occur.

Whilst the original expression might have looked complicated, from a calculus perspective the problem is relatively straight forward. Perhaps surprising that the maximum doesn't occur at the point of inflection on the function, it is easy to put the point on the circle at the point of inflection and see that the curves overlap.

If $f(x)$ is changed to the standard normal curve, the maximum iris area occurs at $x = 0$.

An interesting challenge is for students to dilate the function $f(x)=e^{-x^2}$ so that the maximum iris area occurs when the iris just touches the point of inflection.

If $f(x)=\dfrac{\sqrt{2}\cdot e^{\frac{1}{2}-x^2}}{2}$ then the maximum iris area occurs when the iris just touches the point of inflection.

Students may prefer to see this function written as: $f(x)=\dfrac{\sqrt{2}\cdot e^{\frac{1}{2}}\cdot e^{-x^2}}{2}$ so that it doesn't appear like a translation.

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