NUMB3RS Activity: Set the Trap
Episode: “Provenance”

Topic: Probability Matrices **Grade Level:** 9 - 12
Objective: Students will set up probability matrices and solve problems using iteration.
Time: 15 - 20 minutes
Materials: TI-83 Plus/TI-84 Plus graphing calculator

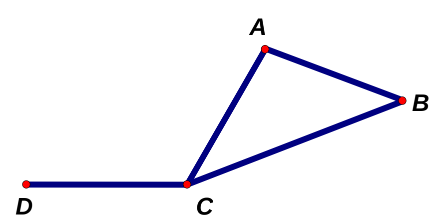
Introduction

In the *NUMB3RS* episode “Provenance,” a famous painting by the artist Camille Pissarro is stolen from a gallery in Los Angeles. Because the painting is well-known, there are a limited number of places it could be shipped. Charlie creates a *network diffusion probability model* to keep track of the possible routes the painting will travel. In the episode, Charlie creates an 18 × 18 **probability matrix** to determine the most likely hiding places for the painting. In this activity, students will learn how to create a probability matrix and use it to calculate the most likely hiding places for the painting.

The activity is split up into three parts: 1) Creating a Probability Matrix, 2) Finding the Most Likely Hiding Place, and 3) Setting the Trap. Part 1 introduces probability matrices and how they are created. A map is provided of the paths the security guard at an art gallery might trace and the students try to find a probability matrix that matches the map. Probability matrices are $n \times n$ matrices that describe the probability the security guard moves to another location depending on his current position. Part 2 explains how to use iteration of a probability matrix on the graphing calculator to find a **steady-state matrix**, which can be used to determine the most likely hiding place for the missing painting. Part 3 is a combination of Parts 1 and 2, where the students are given a map and asked to find the most likely hiding place for the painting.

Discuss with Students

The students are investigating which of the four stations represented by a letter on the graph to the right a security guard is most likely to be visiting. Suppose that a guard is currently visiting station *A* and he has three equally likely choices for where he goes next and randomly chooses one. He can stay at *A*, move to *B*, or move to *C*. There is not a direct path from *A* to *D*, so he cannot go to *D* next. Make sure that you emphasize the assumption that each choice is equally likely is only made to simplify the problem (and in many cases, this assumption is not needed). Another assumption for this graph is that the security guard is always at a station and is never traveling down a hallway. This is not a realistic assumption, but makes it easier to model the problem.



The probability matrix that your students will generate in Question 4 is shown on the right. Review how each entry in the matrix is interpreted in terms of the security guard.

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

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You may want to review the graphing calculator keystrokes for Questions 7 to 9 with your students. The situation for Part 2 refers to the situation of the missing Pissarro. Again, the assumption is that the missing painting is always at a hidden warehouse and is never on one of the paths.

The starting matrix *B* is $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and indicates that the painting initially is being hidden at the Hollywood warehouse. This starting position was chosen at random and students will get the same steady-state matrix if they used the matrix $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ (painting initially hidden in the Glendale warehouse) or $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ (Pasadena warehouse).

Question 7 asks the students to explain the entries for matrix *C* $\begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix}$. You may want to review matrix multiplication to show how these values were calculated. One point of confusion for students may be how time is related to the multiplication operation. Each product of matrix *C* and matrix *A* represents a passage of time. Even though we will assume the painting could be moved every day, the same analysis would apply for any time interval.

The answer to Question 9 shows the probability that the painting is at the Glendale warehouse is $\frac{3}{7}$ (approximately 43%). Don is more likely not to find the painting at the Glendale warehouse (there is a 57% chance it is not at the Glendale warehouse), but if he can only choose one warehouse to set a trap, then the Glendale warehouse would be the best choice. You may want to review this with your students and discuss that these types of calculations only give us a better guess as to the whereabouts of the missing painting, but do not tell us exactly where it is.

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Student Page Answers:
1. *Answers will vary. Some students may argue that station C is the most likely station since it is connected to the greatest number of other stations.* **2.** *B or C. He cannot travel to station D because there is no hallway connecting the stations.* **3a.** $\frac{1}{3}$ **3b.** $\frac{1}{3}$ **3c.** $\frac{1}{3}$ **3d.** *0*
4. *If the guard is currently at station D he has only two equally-likely choices. He can stay at station D or move to station C.*

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

5. *The guard has to be at exactly one of the four stations.* **6.** *Answers will vary. The Glendale warehouse may seem to be the most likely hiding place since it is connected to the greatest number of other warehouses.* **7.** *The thief can keep the painting at the Hollywood warehouse or move it to the Glendale warehouse. Both choices are equally likely. The 0.5 in the first column of the matrix represents the probability of the thief keeping the painting at the Hollywood warehouse. The 0.5 in the second column of the matrix represents the probability of the painting being moved to the Glendale warehouse. The 0 in the third column of the matrix represents the probability of the thief moving the painting to the Pasadena warehouse. This is impossible since the two warehouses are not directly connected.* **8.** *The entries in each successive iteration of the matrix become equal. That is, the entries in the matrix move to a steady state.* **9.** *Over time, the probability the painting will be at the Hollywood warehouse is* $\frac{2}{7}$*, the probability the painting will be at the Glendale warehouse is* $\frac{3}{7}$ *and the probability the painting will be at the Pasadena warehouse is* $\frac{2}{7}$*. The painting is most likely at the Glendale warehouse.* **10.** *Answers will vary. The most likely hiding places may be warehouse I or K since they both have more paths connecting them than the other warehouses.*
11. *The probability matrix is*

$$\begin{array}{cc} & \begin{array}{cccccc} H & I & J & K & L & M \end{array} \\ \begin{array}{c} H \\ I \\ J \\ K \\ L \\ M \end{array} & \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{array}$$

The steady-state matrix becomes $\begin{bmatrix} 0.15 & 0.2 & 0.15 & 0.25 & 0.1 & 0.15 \end{bmatrix}$*. This shows that the most likely hiding place is warehouse K.*

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Name: ________________________________ Date: _______________

NUMB3RS Activity: Set the Trap

In the *NUMB3RS* episode “Provenance,” a famous painting by the artist Camille Pissarro is stolen from a gallery in Los Angeles. Because the painting is well-known there are a limited number of places it could be shipped. Charlie creates a “Network Diffusion Probability Model” to keep track of the possible routes the painting could travel. On the show Charlie creates an 18 × 18 **probability matrix** to determine the most likely hiding places for the painting. In this activity you will learn how to create a probability matrix and use it to calculate the most likely hiding place.

Creating a Probability Matrix

Begin this activity by looking at the theft from the viewpoint of the thief. Assume that the Pissarro is guarded by one security guard. This guard can be at any one of the four stations (*A*, *B*, *C*, *D*) as shown in Figure 1 below. The lines connecting the stations represent the hallways the security guard can travel as he goes from station to station. Note that the security guard can travel from station *A* to station *B* using a direct route, but there is no direct route from station *A* to station *D*. Assume that the security guard randomly decides to either stay at a station or travel to a connected station and each choice is equally likely. If the thief knows which stations the security guard is most likely to be visiting, he can plan his theft with a lower chance of being caught.

1. Which station on the map on the right is the security guard most likely to be visiting? Explain your reasoning.

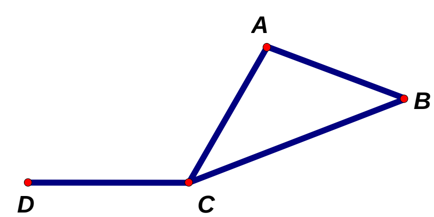

Figure 1: Security Guard Stations

2. If the security guard is currently at station *A*, what stations can he visit next?

3. Determine the probabilities that the security guard will visit each station if he is currently at station *A*.

a. *P*(stays at station *A*)

b. *P*(goes to station *B* next)

c. *P*(goes to station *C* next)

d. *P*(goes to station *D* next)

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4. The probability matrix to the right shows the probabilities that the security guard will visit a particular station when he starts at a particular station. The third row of the matrix shows the probabilities the guard will visit each of the four stations next given that the security guard is currently at station *C*. There are 4 choices: he can go to station *A*, *B*, or *D*, or he can stay at station *C*. The choices are assumed to be equally likely, so the probabilities are each $\frac{1}{4}$. Explain the probabilities for row *D*. Then fill in the missing entries.

Columns
next station visited

Rows
starting station

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

5. One important property of a probability matrix is that the rows sum to 1. Explain why this property is true.

Finding the Most Likely Hiding Place

Assume Don determines that the missing Pissarro is hidden in one of the three warehouses shown on the graph in Figure 2. Assume the thief randomly considers moving the painting among the warehouses every day to avoid being caught. The thief can move the painting from one warehouse to another only if they are connected by an edge on the graph. The thief could also simply leave the painting in the warehouse that day. The probability matrix in Figure 3 matches the graph in Figure 2.

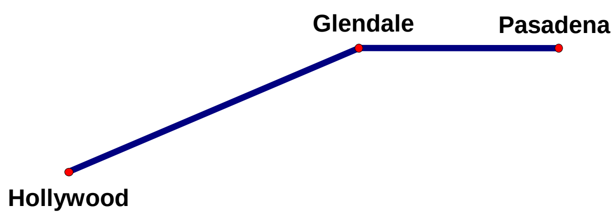

Figure 2: Hidden Pissarro

$$\begin{array}{c} \\ H \\ G \\ P \end{array} \begin{array}{c} \begin{array}{ccc} H & G & P \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Figure 3: Probability Matrix

6. Which warehouse (*H*, *G*, or *P*) will be the most likely place to find the missing Pissarro?

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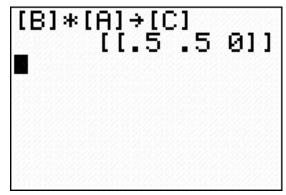
To find the most likely hiding place, enter the probability matrix in your graphing calculator. Press [2nd] [MATRIX], go to the **EDIT** menu, and select **1:[A]**. Key in the entries of the matrix as shown at the right. Note that the calculator converts fractions into decimals when you press [ENTER].

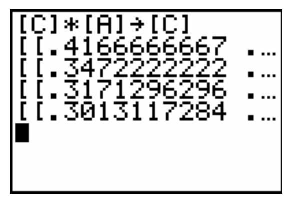
Assume that the thief initially hid the painting in warehouse *H*. Matrix *B* can be created to represent this situation. Because we know the painting is in warehouse *H* there is a '1' in the first column. The second column in matrix *B* is 0, and represents the probability the painting is in warehouse *G*. The third column is also 0. Note that the row sum of this matrix is 1.

Use your calculator to multiply matrix *B* and matrix *A*, and store the result as matrix *C*.

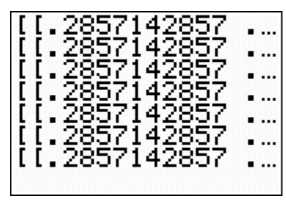


7. Explain what matrix *C* tells you about the location of the painting after the first day that it could have been moved. Explain why the row sum of this matrix is 1.

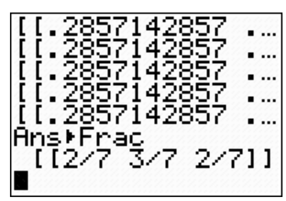
Now multiply matrix *C* and matrix *A* as shown at the right, and store the result as matrix *C*. This new matrix *C* describes the probabilities of the various hiding places after day 2. This iteration process can be continued each time you press [ENTER].



8. Keep pressing [ENTER] until a pattern seems to emerge. Explain what you see. Why do you think this is happening?



9. If you choose **1:åFrac** under the [MATH] menu you can see the **steady-state matrix** represented by fractions. Explain what each fraction represents. According to this result, which warehouse is most likely to contain the missing painting?



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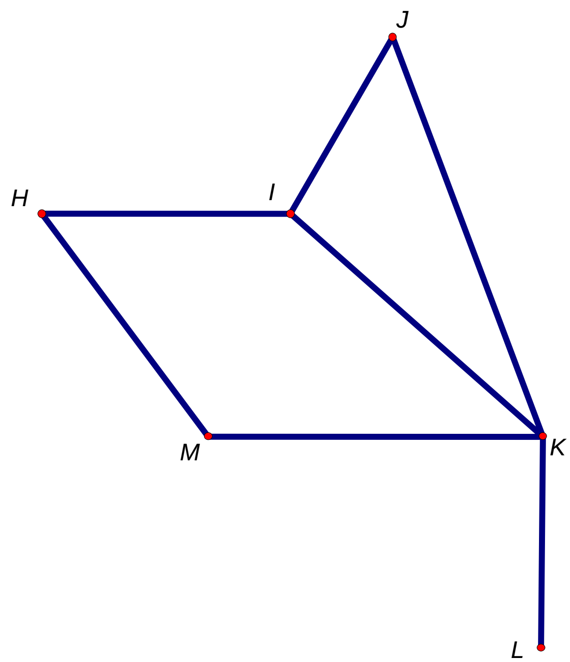
Set the Trap

The thief hid the Pissarro painting in one of the warehouses represented as vertices on the graph below. Assume the thief has been moving the painting around at random and can only transport the painting among the warehouses if the warehouses are connected by an edge.



10. If you were Don and could only search one warehouse, which one would you choose? Explain your choice.

11. Create a probability matrix that represents this situation and use it to determine a steady-state matrix. Use the steady-state matrix to determine the most likely hiding place for the Pissarro.

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The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

Probability Matrices

- An introduction to probability matrices and a step by step example is provided at **http://mathforum.org/library/drmath/view/52192.html**.

- An introduction a special probability matrix called a "Brand-Switching" matrix is provided at **http://mathforum.org/library/drmath/view/54265.html**.

- The steady-state matrix in Question 9 of this activity was determined using iteration and solved the problem introduced in Question 6. Try to find the same solution you found in Question 9 by solving the linear system of equations generated by the following matrix multiplication problem. Make sure one of your equations state that the sum of *a*, *b*, and *c* equals 1.

$$\begin{bmatrix} a & b & c \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}$$

  Explain why this method works.

- You may have noticed that the vertex with the greatest number of edges connected to it was the most likely hiding place for the Pissarro. Create a few of your own maps and try to determine if this claim is always true.

- Find the steady-state matrix that corresponds to the security guard problem introduced in Questions 1 to 5.

- An assumption in the previous question is that all choices for the next station visited are equally likely. Assume that the choices for the next station are not equally likely. Try to find a probability matrix with a steady-state solution that would not give the same solution as in the question above. Remember to make sure that the row sums of the probability matrix equal 1.

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