EXPLORATIONS

Activity 11

Get Your Numbers in Shape

Objectives

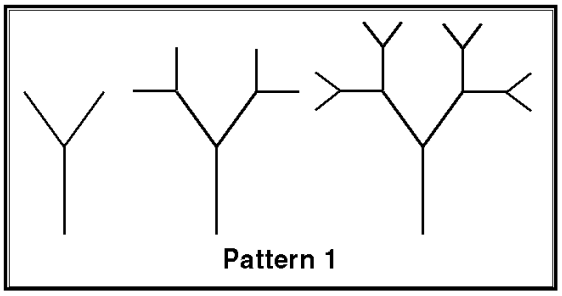
- To use technology to explore patterns
- To use inductive reasoning to make conjectures about patterns
- To use technology to produce a sequence
- To find the *x* value of a function, given the *y* value
- To find the *y* value of a function, given the *x* value
- To find a linear or a quadratic equation for a given pattern

Materials

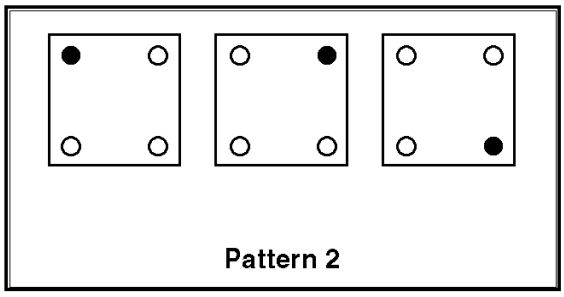
- TI-83 Plus
- Small cubes or candy

Introduction

Study the pictures below and find the next shape in each of the patterns shown.



Pattern 1



Pattern 2

The process that you have used to find the next shape in each pattern is called *inductive reasoning*. You have used this type of reasoning since you were a baby. You learned how to eat using the proper utensils and how to turn on the television by observing others and drawing conclusions. After several trials, you perfected your skills. Inductive reasoning allows you to make generalizations based on a pattern of specific examples or past events. These generalizations are called *conjectures* or *hypotheses*. Mathematicians and scientists use inductive reasoning to make discoveries and develop formulas based on their discoveries.

Mathematicians have used number patterns to describe a variety of phenomena in nature. For example, a famous pattern that is found throughout nature is the *Fibonacci sequence*. You will investigate a variety of sequences in this activity.

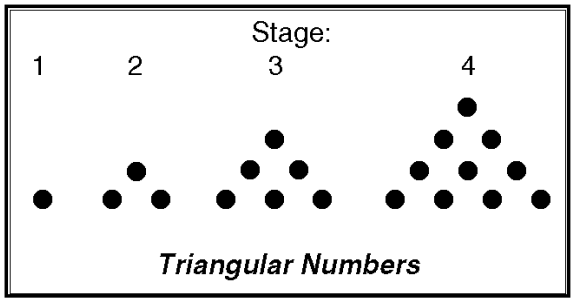
Problem

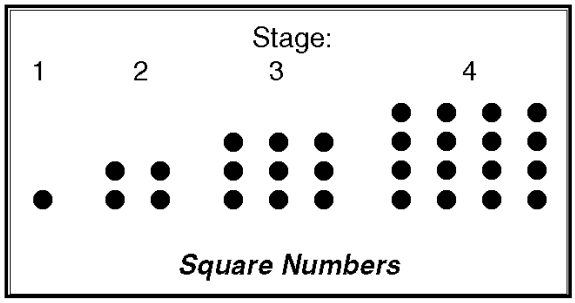
How do you use inductive reasoning to find a formula for the patterns that describe an array of objects or a sequence of numbers?

Collecting the data — Part I

A *triangular number* is a number that can be represented by a triangular arrangement of objects.

A *square number* is a number that can be represented by a square arrangement of objects. Numbers that correspond to geometric figures are called *figurate numbers*. The diagrams below illustrate triangular numbers and square numbers.





Use the objects that your teacher will give you to find the next three stages for the triangular numbers. Complete the table on the **Data Collection and Analysis** page.

Do you notice a pattern in the table? Suppose you needed to know the 100th triangular number. You could continue to form the pattern using your objects, however this could become tedious and time-consuming. If you knew a rule or *function* for calculating any term in this sequence, you could find the 100th term of the sequence with relative ease.

You will use the *finite differences* technique to find a function for this sequence. This method requires that you find the difference between successive values in the sequence. If the first differences are equal, then the function or rule that describes the sequence is *linear.* If the second differences are equal, then the function or rule that describes the sequence is *quadratic*. You can use the TI-83 Plus to find the differences.

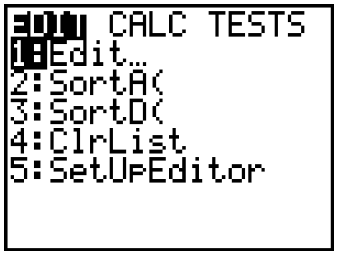
Setting up the TI-83 Plus

Before starting your data collection, make sure that the TI-83 Plus has the STAT PLOTS turned OFF, Y= functions turned OFF or cleared, the MODE and FORMAT set to their defaults, and the lists cleared. See the Appendix for a detailed description of the general setup steps.

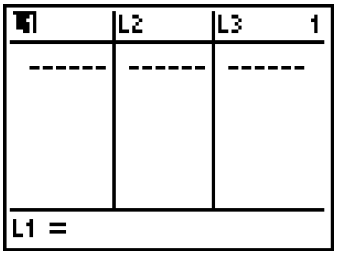
Entering the data in the TI-83 Plus

Enter the stage number data in **L1** and the value in **L2**. Since the stage numbers form a sequence, you will use the *sequence* command on the TI-83 Plus to enter the data in **L1**.

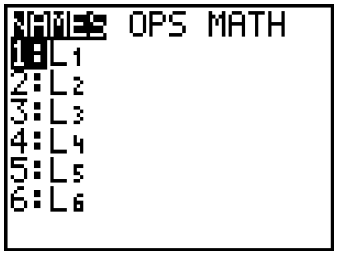
1. Press [STAT] and select **1:Edit** by pressing [ENTER].



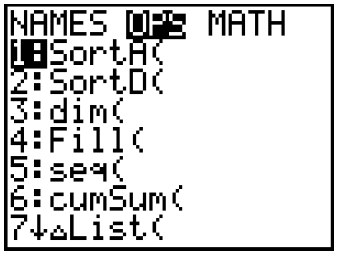
2. Press [▲] to highlight **L1**.



3. Press [2nd] [LIST] to display the **NAMES** menu.

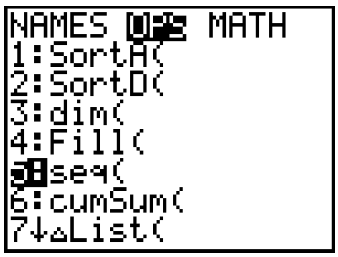


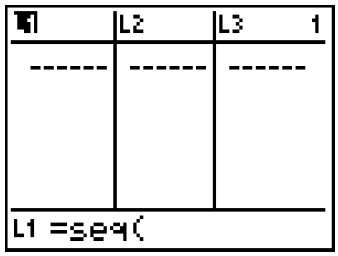
4. Move the cursor to the **OPS** menu.



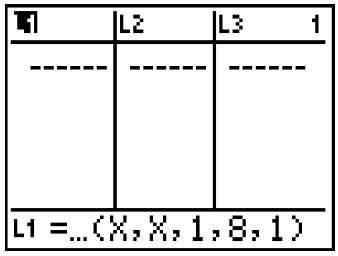
5. Select **5:seq(** and press [ENTER] for the sequence command.

The components of the sequence command are **seq(**expression or formula, variable, beginning value, ending value, increment).

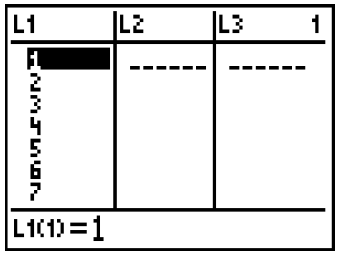




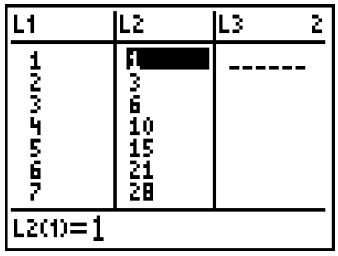
6. Press [X,T,Θ,*n*] [,] [X,T,Θ,*n*] [,] **1** [,] **8** [,] **1** [)] to enter the command for finding a sequence of consecutive integers from 1 to 8.



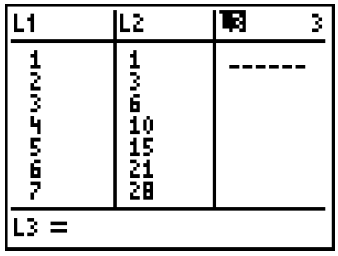
7. Press [ENTER].



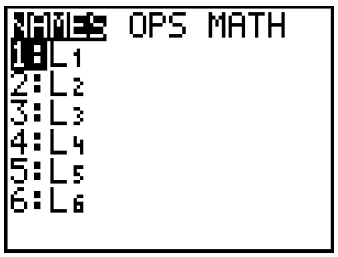
8. Enter the values in **L2**.



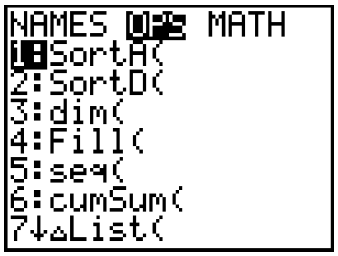
9. Move the cursor to the top of **L3** to find the first difference between the successive terms in **L2**.



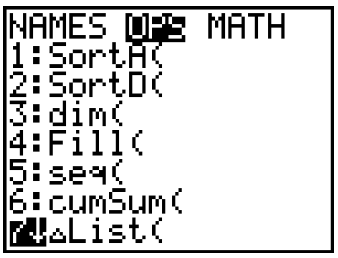
10. Press 2nd [LIST] to display the **NAMES** menu.



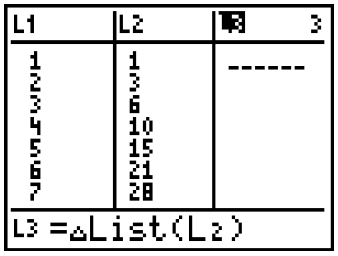
11. Move the cursor to the **OPS** menu.



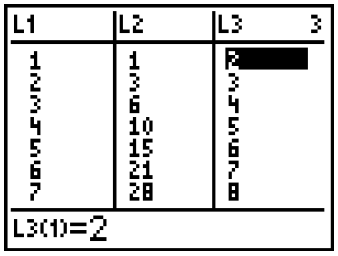
12. Select **7:ΔList(** and press ENTER.



13. Press 2nd [L2]).



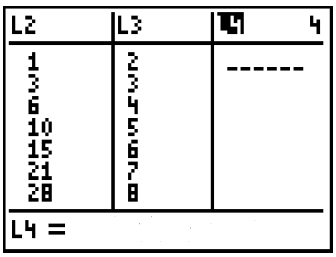
14. Press ENTER to see the difference between the successive terms in **L2**.



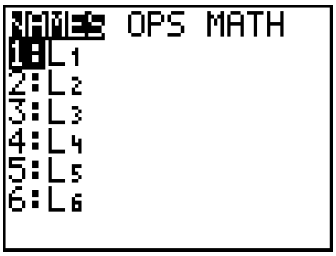
The numbers in **L3** represent the first difference of the sequence. Observe that the values are not equal. Therefore, the function that describes this sequence is not *linear*.

Term Number	1	2	3	4	5	6	7	8
Value	1	3	6	10	15	21	28	36
		2	3	4	5	6	7	8

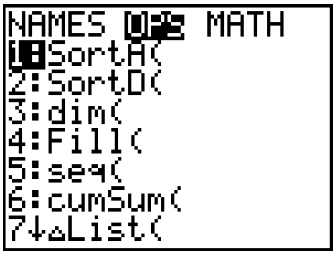
15. Move the cursor to the top of **L4** to find the second difference between the successive terms in **L3**.



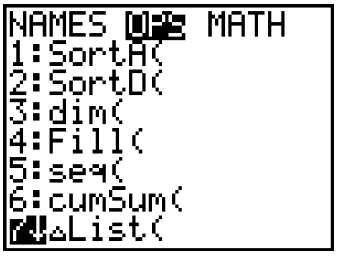
16. Press [2nd] [LIST] to display the **NAMES** menu.



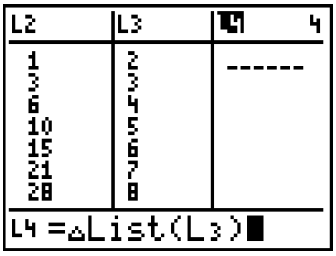
17. Move the cursor to the **OPS** menu.



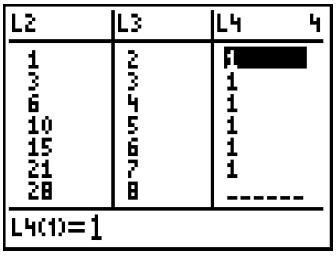
18. Select **7:ΔList(** and press [ENTER].



19. Press [2nd] [L3] [)].



20. Press [ENTER] to see the difference between the successive terms in **L3**.

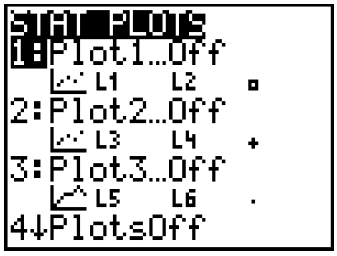


Since the second differences are equal, the function that describes this sequence is *quadratic.*

Graphing the data: Setting up a scatter plot

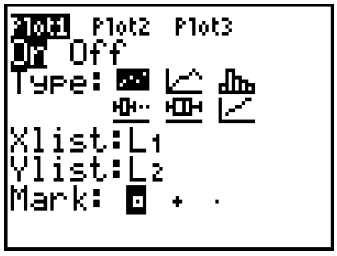
Plot the data using **L1** and **L2**.

1. Press [2nd] [STAT PLOT] and select **1:Plot1** by pressing [ENTER].



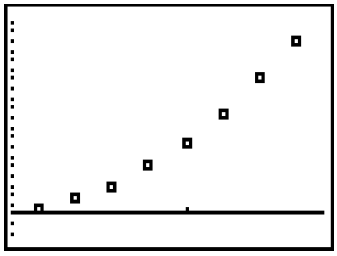
2. Set up the plot by pressing [ENTER] [▼] [ENTER] [▼] [2nd] [L1] [ENTER] [2nd] [L2] [ENTER] [ENTER].

Note: *Press* [▼] [▼] *if* **L1** *and* **L2** *are already displayed.*



3. Press [ZOOM].

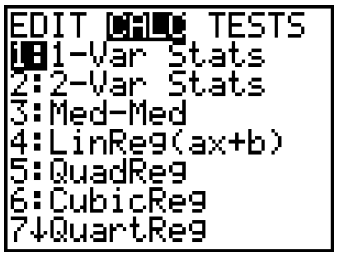
4. Press the up arrow [▲] or the down arrow [▼] to select **9:ZoomStat** and press [ENTER].



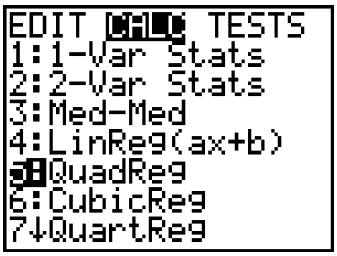
Analyzing the data: Finding a trend line

Use a quadratic regression to find the quadratic function that describes this sequence.

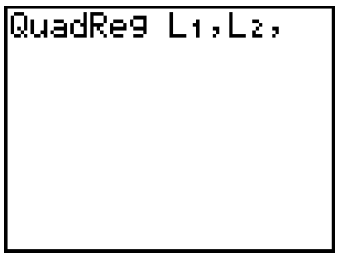
1. Press [STAT] and move the cursor to the **CALC** menu.



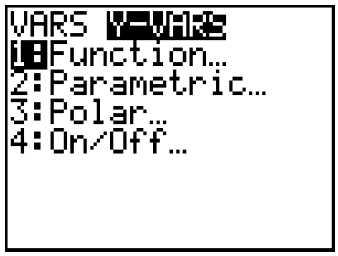
2. Select **5:QuadReg** and press [ENTER].



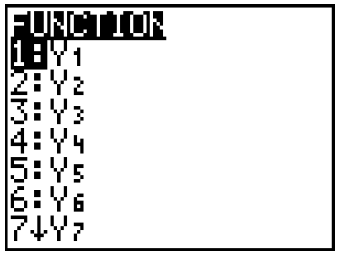
3. Press 2nd [L1] , 2nd [L2] ,.



4. Press VARS and move the cursor to the **Y-VARS** menu.



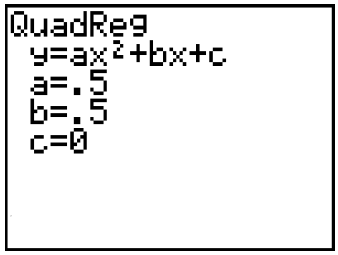
5. Select **1:Function** by pressing ENTER.



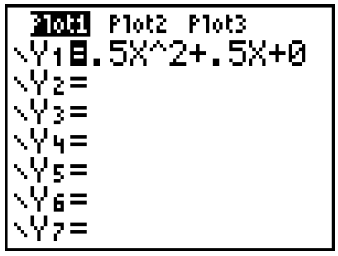
6. Select **1:Y1** by pressing ENTER.

7. Press ENTER to see the function.

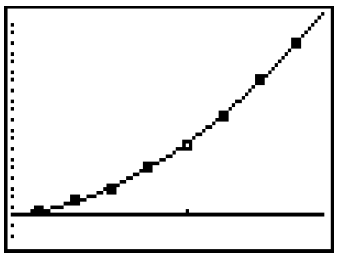
The function that describes the sequence for triangular numbers has been stored in **Y1**.



8. Press Y= to see the equation.



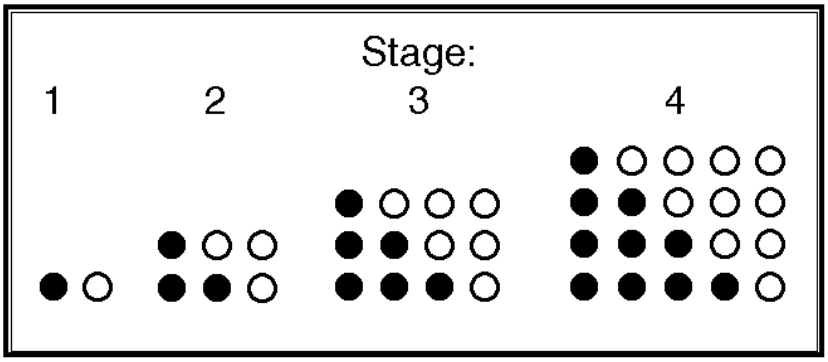
9. Press GRAPH to see the graph of the function.



Use the function to answer Part I questions 1 through 4 on the **Data Collection and Analysis** page.

Collecting the data — Part II

You can find a formula for the *triangular numbers* by finding the differences and then factoring to find a pattern. Examine the diagram and table below for triangular numbers. The black dots represent the triangular number pattern.



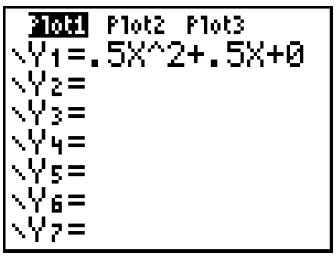
1. Term Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	***n***
2. Value	1	3	6	10	15	21	28	36	
3. Twice the Value	2	6	12	20	30	42	56	72	
4. Factors for Row 3	**1** • 2	**2** • 3	**3** • 4	**4** • 5	**5** • 6	**6** • 7	**7** • 8	**8** • 9	***n***(*n* + 1)

Since the formula represents twice the actual triangular numbers, you must divide by 2 to find the formula for the triangular numbers. The formula is: $y = \frac{1}{2}\, n(n + 1)$ or $y = .5\, n(n + 1)$.

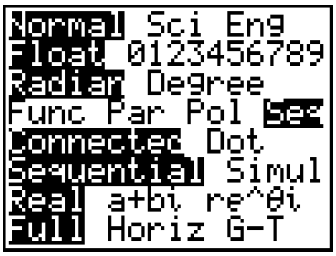
The formula for triangular numbers is a sequence. A *sequence* is an ordered list of numbers, specifically integers. You have graphed the formula in function mode. In function mode, values such as 1.5, 2.3, and 4.7 can be calculated. For a sequence, however, the domain and range are integers and do not include rational numbers. Therefore, you should graph the sequence as a set of points that are not connected.

You will develop two types of sequence formulas, a recursive formula and an explicit formula. A *recursive formula* is a formula that tells how to find a term using the previous term. An *explicit formula* is a formula that tells how to find the value of any term of a sequence without finding all the previous terms.

1. Turn off the equation to graph in sequence mode. Press [Y=] [◄] [ENTER] [▼] [►].



2. Press [MODE] [▼] [▼] [▼] [►] [►] [►] [ENTER] to change the mode to sequence mode.



Develop a recursive formula for the sequence. The first value that you must enter is *n*Min. This is the smallest *n* to be evaluated (where *n* is the term number).

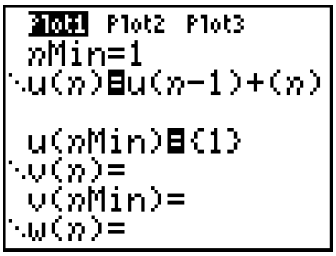
The next value that you must enter is the recursive formula. For the triangular numbers the formula is: u(*n*) = u(*n*-1) + (*n*). Use the value of the previous term plus the term number to get the next term in the sequence. For example, to produce *term 4* from *term 3*, you add *6 + 4 = 10*.

Note: *6 is the value of the* 3rd *term and* 4 *is the term number.*

3. Press [Y=] to set up the function.

4. Press [2nd] [U] [(] [X,T,Θ,*n*] [−] **1** [)] [+] [(] [X,T,Θ,*n*] [)] to enter the formula in *u(n)*.

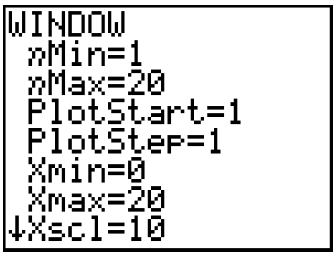
The last value that you must enter is the first value in the sequence. The first value is **1** for the triangular numbers.

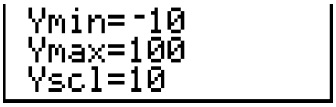


5. Move the cursor to u(*n*Min).

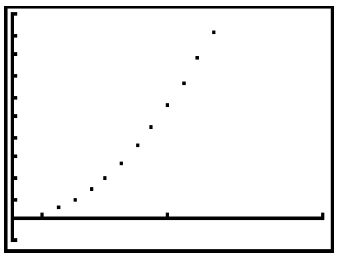
6. Press [(] **1** [)].

7. Press [WINDOW] and set the window values as shown.



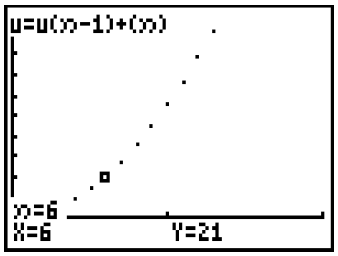


8. Turn off the plot to see the graph of the sequence. Press 2nd [STAT PLOT] ENTER ▶ ENTER.

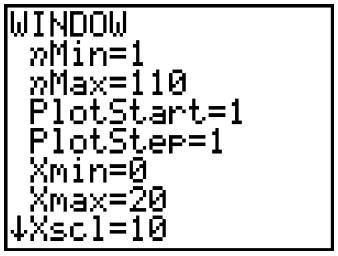


9. Press GRAPH to see the plot.

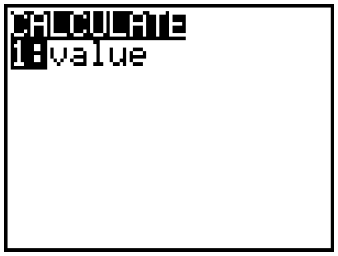
10. Press TRACE and then press ▶ to see the values of the terms for the sequence.

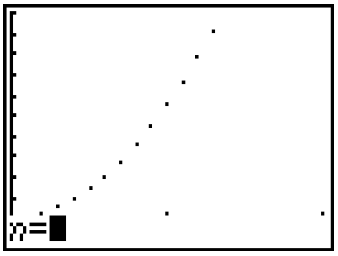


11. Use the formula to find the value of the 100th term. Press WINDOW ▼ **1 1 0** ENTER to change the *n*Max value.

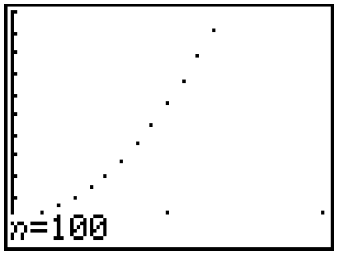


12. Press 2nd [CALC] and press ENTER.

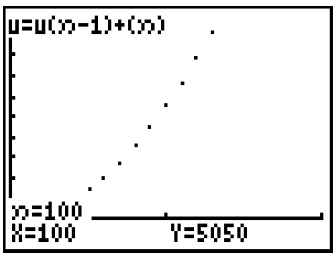




13. Type **1 0 0**.



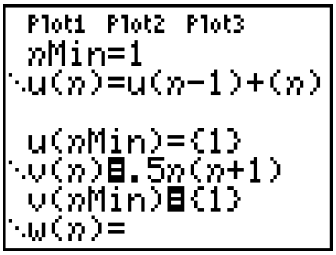
14. Press ENTER.



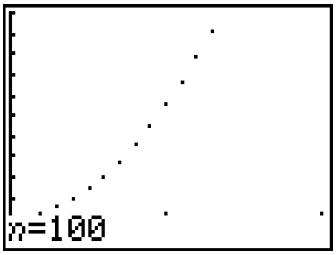
Develop an explicit formula for the sequence. The explicit formula is: *v(n) = .5n(n + 1)*.

15. Turn off the recursive formula. Press Y= ◀ ENTER ▼ ▶.

16. Press . **5** X,T,Θ,*n* (X,T,Θ,*n* + **1**) ENTER (**1**) ENTER to enter the formula in the TI-83 Plus.



17. Press GRAPH to see the graph of the sequence.



Follow the instructions in step 11 to find the 75th term.

Find a recursive formula and an explicit formula for the *rectangular numbers* and the *pentagonal numbers* in Part II questions 1 and 2 on the **Data Collection and Analysis** page.

Data Collection and Analysis

Name _____

Date _____

Activity 11: Get Your Numbers in Shape

Collecting the data

Record your data for triangular numbers in the table below.

Stage Number	1	2	3	4	5	6	7	8
Value	1	3	6	10				

Analyzing the data — Part I

Use your equation from number 8 in the **Analyzing the data: Finding a trend line** section to answer questions 1 through 4.

1. Find the 20th *triangular number*. _____

2. Find the 100th *triangular number*. _____

3. What is the stage for a *triangular number* that equals 120? _____

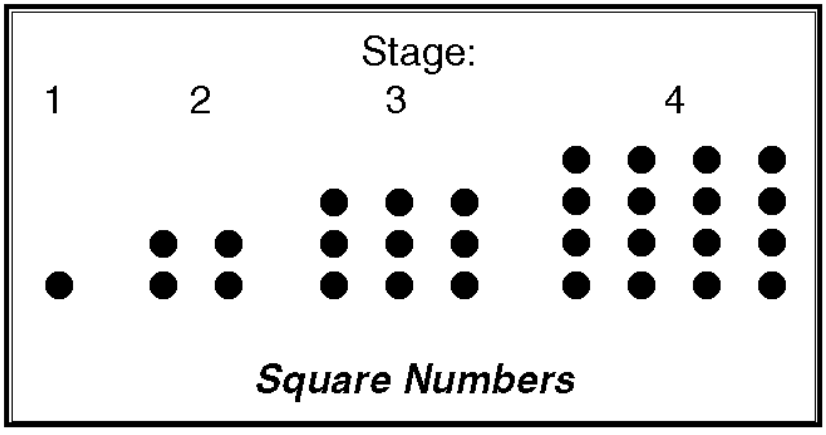
4. What is the stage for a *triangular number* that equals 496? _____

5. Find a function for the *square numbers*. _____

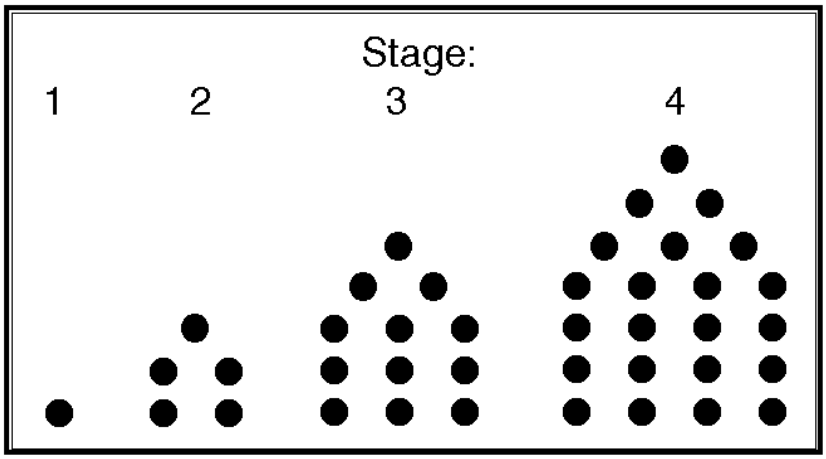
Analyzing the data — Part II

1. The pattern below represents *rectangular numbers*. Find a function for this pattern using *finite differences*.

_____



2. The pattern below represents *pentagonal numbers*. Find a function for this pattern using *finite differences*.



3. You go to a back-to-school party. Thirty of your friends are at the party. After the long summer break, you are happy to see all of your friends. You begin shaking hands with each person at the party. In fact, each person at the party shakes hands with everyone else at the party. How many handshakes are there altogether?

Teacher Notes

Activity 11

Get Your Numbers in Shape

Objectives

- To use technology to explore patterns
- To use inductive reasoning to make conjectures about patterns
- To use technology to produce a sequence
- To find the *x* value of a function, given the *y* value
- To find the *y* value of a function, given the *x* value
- To find a linear or a quadratic equation for a given pattern

Materials

- TI-83 Plus
- Small cubes or candy

Preparation

- For question number 7, give the students a hint if you are doing the **Collecting the data — Part II** section. **Hint:** Look at the dots. What two figurate patterns form the pentagonal pattern? Look at the formulas for these patterns.

Management

- For question number 8, have one group of students act out the situation with a smaller group of students. Develop a table with a smaller number of people. Gradually increase the number of students until they can make a table for 20 students. Ask the students to draw a polygon, where the vertices of the polygon represent the students.

Answers to Data Collection and Analysis

Collecting the data

- Sample data

Term Number	1	2	3	4	5	6	7	8
Value	1	3	6	10	15	21	28	36

2 3 4 5 6 7 8

Analyzing the data — Part I

Use your equation from number 8 in the **Analyzing the data: Finding a trend line** section to answer questions 1 through 4.

1. Find the 20th *triangular number*. *210*

2. Find the 100th *triangular number*. *5050*

3. What is the stage for a *triangular number* that equals 120? *15*

4. What is the stage for a *triangular number* that equals 496? *31*

5. Find a function for the *square numbers*. $y = x^2$

Analyzing the data — Part II

1. The pattern below represents *rectangular numbers*. Find a function for this pattern using *finite differences*. $n(n + 1)$

2. The pattern below represents *pentagonal numbers*. Find a function for this pattern using *finite differences*. $y = 1.5n^2 - .5n$

3. You go to a back-to-school party. Thirty of your friends are at the party. After the long summer break, you are happy to see all of your friends. You begin shaking hands with each person at the party. In fact, each person at the party shakes hands with everyone else at the party. How many handshakes are there altogether? *formula:* $y = \frac{1}{2} n(n - 1)$

For 31 people there are 465 handshakes.