Hot Tub

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Abstract: This activity is an application of integration. Students first use the calculator to determine specific numeric results to enable them to understand the dynamics of the problem. They then use the symbolic capacity of their calculator to find the maximum volume. This activity emphasizes using the integral of a rate of change to give the accumulated change. The definite integral of the rate of change of a quantity is interpreted as the change of the quantity.

NCTM Principles and Standards:

Algebra standards

a) Understand patterns, relations, and functions
b) generalize patterns using explicitly defined and recursively defined functions;
c) analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
d) use symbolic algebra to represent and explain mathematical relationships;
e) judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
f) draw reasonable conclusions about a situation being modeled.

Geometry standards:

a) Analyze characteristics and properties of two- and three-dimensional geometric shapes and mathematical about geometric relationships
b) draw and construct representations of two- three-dimensional geometric objects using a variety of tools;

Problem Solving Standard build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof Standard

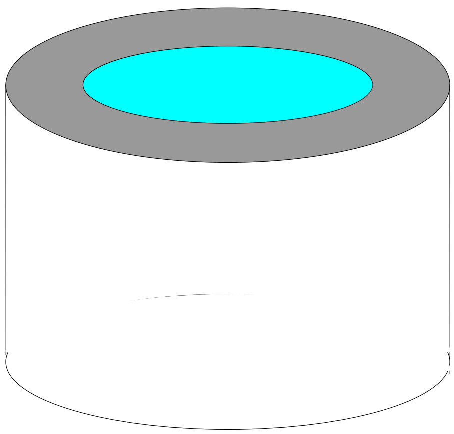
a) recognize reasoning and proof as fundamental aspects of mathematics;
b) make and investigate mathematical conjectures;
c) develop and evaluate mathematical arguments and proofs;
d) select and use various types of reasoning and methods of proof.

Representation Standard : use representations to model and interpret physical, social, and phenomena.

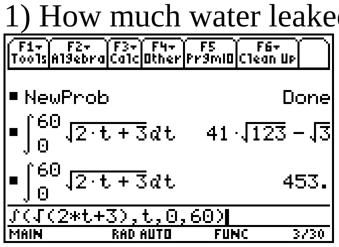
Key topic: Applications of Definite Integrals, Fundamental Theorem of Calculus

Degree of Difficulty: Elementary to moderate
Needed Materials: TI-89 calculators

Situation: At the beach last summer, we rented a house with a hot tub. As soon as I turned it on, though, it started to leak and I decided to try to fill it with a hose. Suppose that the tub was leaking at the rate of $\sqrt{2t+3}$ gallons per minute, that the rate of water from the hose was 6 gallons per minute, and that the tub initially had 400 gallons of water in it.



1) How much water leaked out of the tub during the first hour? (answer:



2) How much water was added to the tub in the first hour? (answer: 60*6 = 360 gallons)

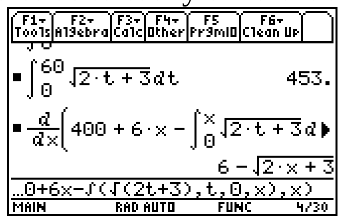
3) How much water was in the tub at the end of the first hour? (answer:
$400 - 453 + 360 = 307$ *gallons*

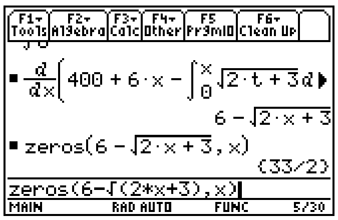
4) What was the largest amount of water in the tub during the first hour?

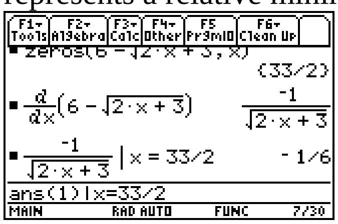
To determine this we first need to get an expression for the amount of water in the tub at any particular time.

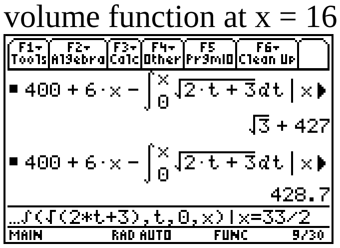
- we start off with 400 gallons
- we are adding water at the rate of 6 gallons/minute – so at the end of x minutes we've added 6x gallons

the tub is leaking water at the rate of $\sqrt{2t+3}$ gallons per minute so after x minutes it has leaked $\int_0^x \sqrt{2t+3}\,dt$ gallons. We are using the integral of a rate of change to give the accumulated change. The definite integral of the rate of change of a quantity over the interval (0, x) is the change of the quantity over the interval (0, x).

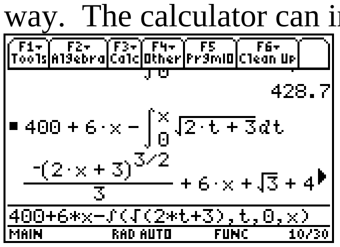
- So the amount at time x is: $400 + 6x - \int_0^x \sqrt{2t+3}\,dt$
- To find the maximum amount, we need to differentiate this quantity with respect to x and find where this derivative equals zero which will give us the critical value. The fundamental theorem of calculus enables us to easily differentiate the definite integral or we can have the calculator do this: 



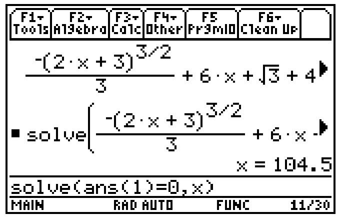
- The critical value occurs at 16 ½ minutes. To find out whether this critical value represents a relative minimum or maximum, we can examine the second derivative:
- 

Since the second the derivative is negative, we have relative maximum when x = 16 ½ minutes.
- How much water will the tub have then? To find out, we find the value of our volume function at x = 16 ½ minutes:
- 

- The maximum amount of water in the tub is 428.7 gallons

We could have used the features of the calculator to do this problem in a slightly different way. The calculator can integrate our volume function directly:


and we could have used this expression in our calculations.

5) When will the tub ever run out of water? To find out, we can set the amount of water equal to zero and solve for x: 

- So, after just over an hour and a half, the tub will be empty!