Thursday Night PreCalculus, September 28, 2023

Rational Functions: Zeros, Holes, Vertical Asymptotes, and End Behavior

Problems

1. Find the location of any zeros and holes for the graph of the given rational function.

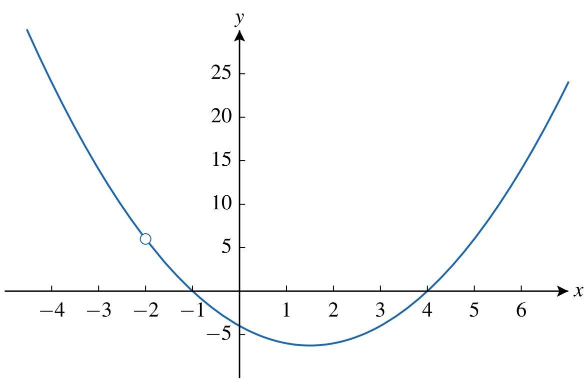
(a) $f(x) = \dfrac{x^3 - x^2 - 10x - 8}{x + 2}$

$x^3 - x^2 - 10x - 8 = (x + 2)(x - 4)(x + 1)$ Synthetic division

$$f(x) = \frac{(x + 2)(x - 4)(x + 1)}{x + 2}$$

Hole: $x = -2$

Zeros: $x = 4, \ -1$



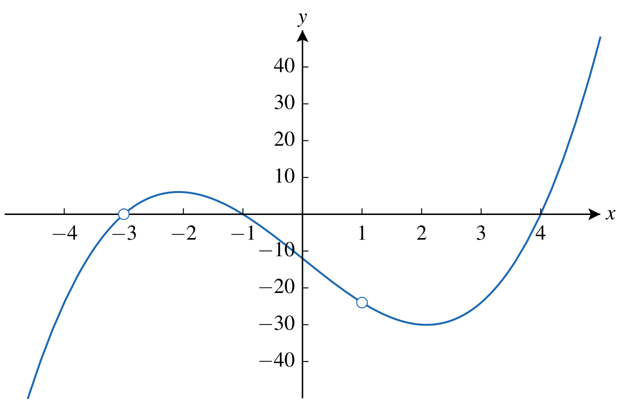
Note: Synthetic division is not in the AP Precalculus course, nor is factoring a cubic with four terms. The only factoring required involves a common factor and rules of quadratics.

(b) $f(x) = \dfrac{(x+3)^2(x+1)(x-1)(x-4)}{x^2+2x-3}$

$$f(x) = \frac{(x+3)^2(x+1)(x-1)(x-4)}{x^2+2x-3} = \frac{(x+3)^2(x+1)(x-1)(x-4)}{(x+3)(x-1)}$$

Holes: $x = -3,\ 1$

Zeros: $x = -1,\ 4$



2. Find the vertical asymptotes for the graph of the given rational function and sketch a complete graph.

(a) $f(x) = \dfrac{x^3 - 5x^2 + 6x}{x^2 - 9}$

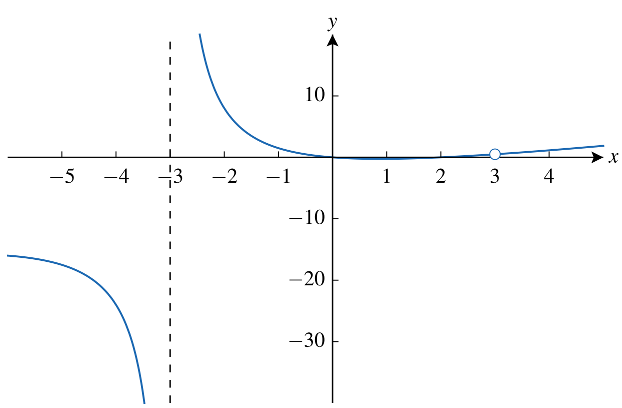
$$f(x) = \frac{x^3 - 5x^2 + 6x}{x^2 - 9} = \frac{x(x^2 - 5x + 6)}{x^2 - 9} = \frac{x(x-3)(x-2)}{(x-3)(x+3)}$$

VA: $x = -3$

Hole: $x = 3$

As $x \to -3^-$: $\dfrac{x(x-3)(x-2)}{(x-3)(x+3)} = \dfrac{x(x-2)}{x+3} = \dfrac{(-3)(-5)}{(-)} \to -\infty$

As $x \to -3^+$: $\dfrac{x(x-3)(x-2)}{(x-3)(x+3)} = \dfrac{x(x-2)}{x+3} = \dfrac{(-3)(-5)}{(+)} \to +\infty$



(b) $f(x) = \dfrac{x^2 - 4}{(x-2)(x^2 - 6x + 5)}$

$$f(x) = \frac{x^2 - 4}{(x-2)(x^2 - 6x + 5)} = \frac{(x-2)(x+2)}{(x-2)(x-5)(x-1)}$$

VA: $x = 1,\ 5$

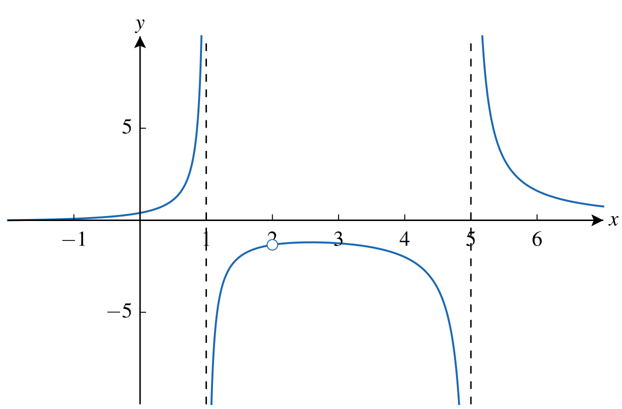
Hole: $x = 2$

As $x \to 1^-$: $\dfrac{(x-2)(x+2)}{(x-2)(x-5)(x-1)} = \dfrac{x+2}{(x-5)(x-1)} = \dfrac{(3)}{(-4)(-)} \to \infty$

As $x \to 1^+$: $\dfrac{(x-2)(x+2)}{(x-2)(x-5)(x-1)} = \dfrac{x+2}{(x-5)(x-1)} = \dfrac{(3)}{(-4)(+)} \to -\infty$

As $x \to 5^-$: $\dfrac{(x-2)(x+2)}{(x-2)(x-5)(x-1)} = \dfrac{x+2}{(x-5)(x-1)} = \dfrac{(7)}{(-)(4)} \to -\infty$

As $x \to 5^+$: $\dfrac{(x-2)(x+2)}{(x-2)(x-5)(x-1)} = \dfrac{x+2}{(x-5)(x-1)} = \dfrac{(7)}{(+)(4)} \to +\infty$



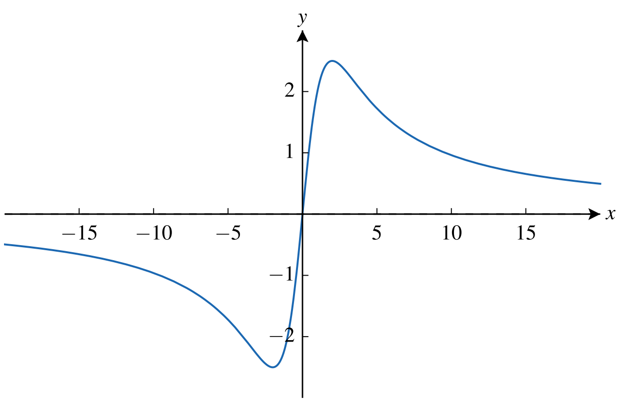
3. Express the end behavior of each rational function using limit notation and sketch a complete graph.

(a) $f(x) = \dfrac{10x}{x^2+4}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{10x}{x^2+4} = 0$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{10x}{x^2+4} = 0$$

Holes? VA? Zeros?



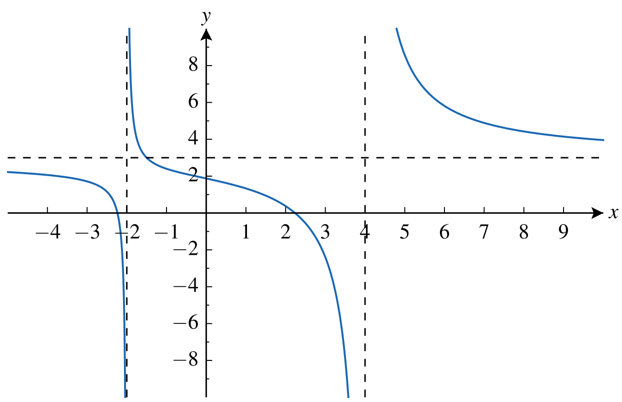
(b) $f(x) = \dfrac{3x^2 - 15}{x^2 - 2x - 8}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x^2 - 15}{x^2 - 2x - 8} = 3$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2 - 15}{x^2 - 2x - 8} = 3$$

$$f(x) = \frac{3x^2 - 15}{x^2 - 2x - 8} = \frac{3(x^2 - 5)}{(x-4)(x+2)} = \frac{3(x - \sqrt{5})(x + \sqrt{5})}{(x-4)(x+2)}$$

Holes? VA? Zeros?



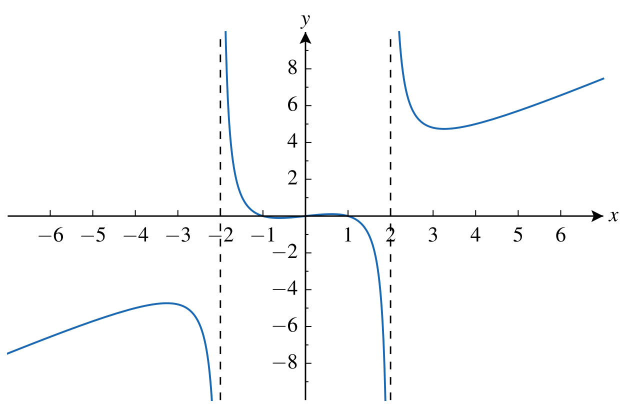
(c) $f(x) = \dfrac{x^3 - x}{x^2 - 4}$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^3 - x}{x^2 - 4} = -\infty$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^3 - x}{x^2 - 4} = \infty$$

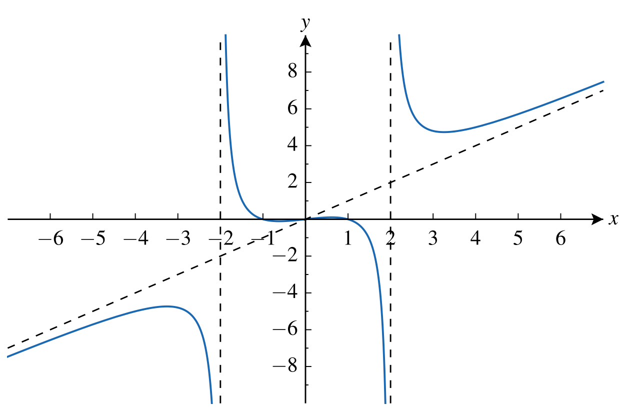
$$f(x) = \frac{x^3 - x}{x^2 - 4} = \frac{x(x^2 - 1)}{x^2 - 4} = \frac{x(x-1)(x+1)}{(x-2)(x+2)}$$

Holes?   VA?   Zeros?



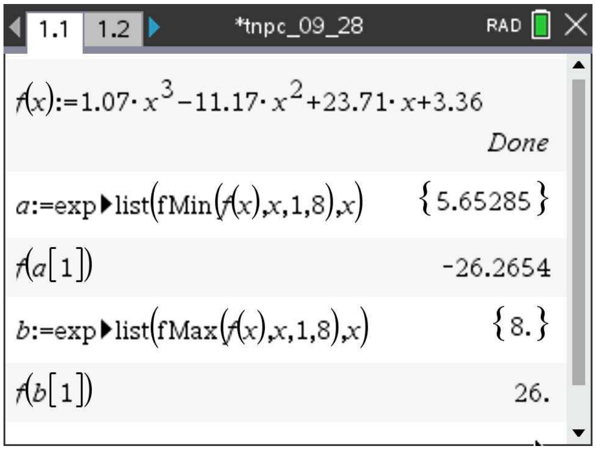
Note:

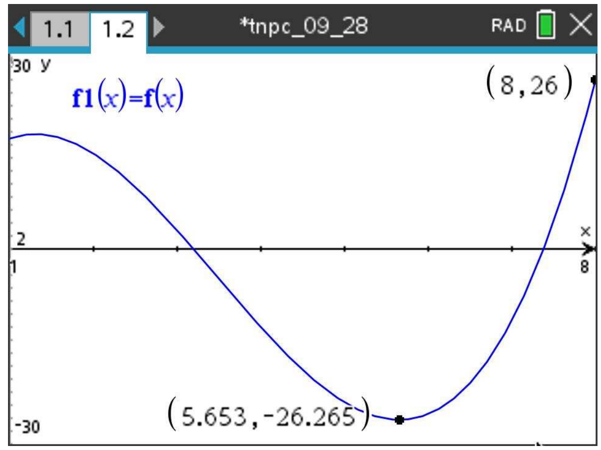
$$f(x) = \frac{x^3 - x}{x^2 - 4} = x + \frac{3x}{x^2 - 4}$$



Overtime Problems

1. The function f is given by $f(x) = 1.07x^3 - 11.17x^2 + 23.71x + 3.36$. Find the absolute extreme values of f on the closed interval $-1 \leq x \leq 8$.



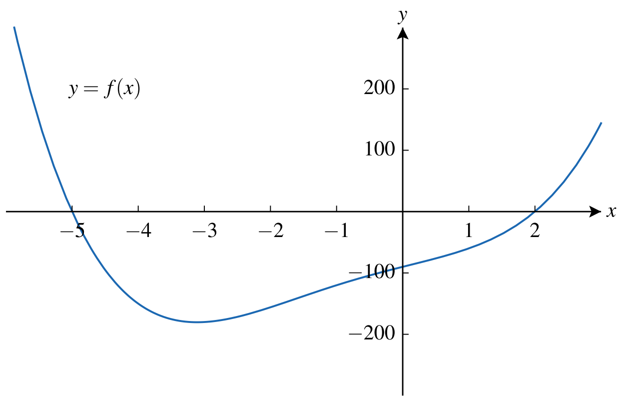


The minimum value is $f(5.653) = -26.265$.

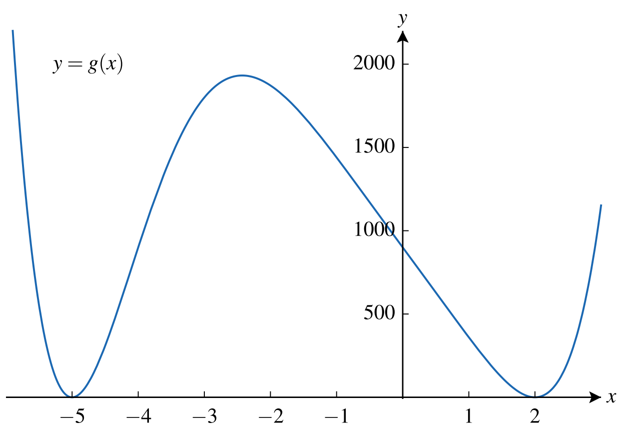
The maximum value is $f(8) = 26$

2. A polynomial function f of degree n has zeros at $x = 2$, $x = -5$, and $x = 3i$. If $f(x) \geq 0$ for all real values of x, what is the value of n?

$$f(x) = (x-2)(x+5)(x-3i)(x+3i) = (x-2)(x+5)(x^2+9)$$

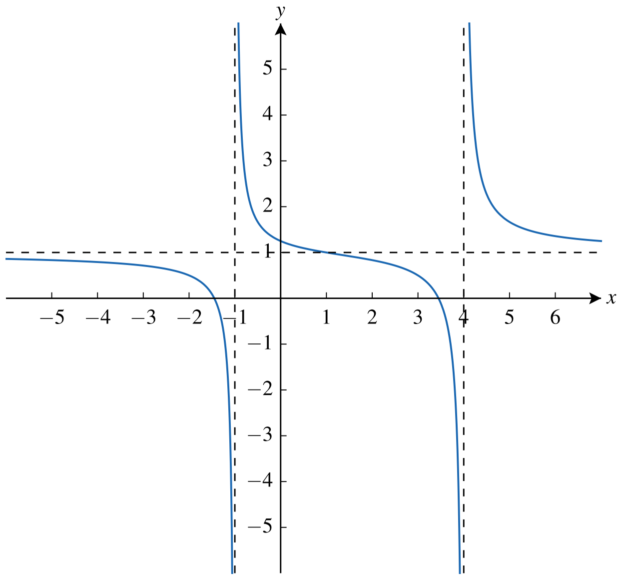


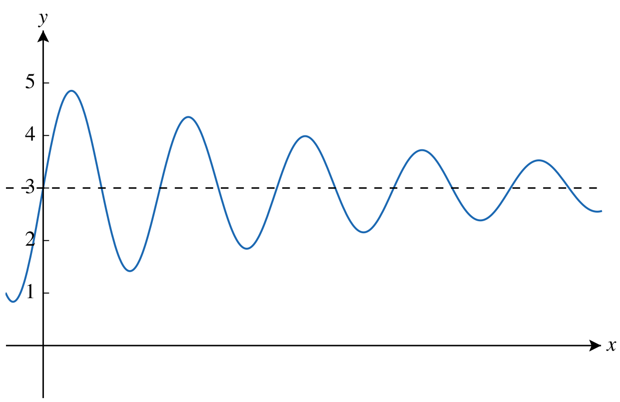
$$g(x) = (x-2)^2(x+5)^2(x^2+9)$$



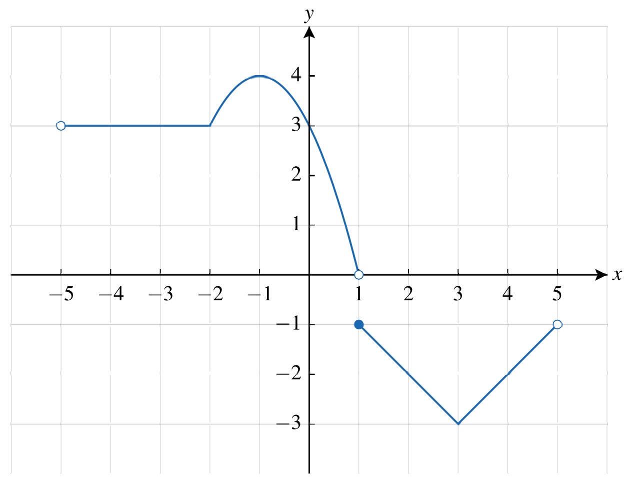
$n = 6$

3. Can the graph of a function f cross its horizontal asymptote(s)?





4. The graph of the function f is shown in the figure.



Find the interval(s) on which the function f is decreasing.

Defintion

A function f is **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$

The function f is **decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$

Solution

The function f is decreasing on the interval $[-1, 3]$.