NUMB3RS Activity: Guess My Type or Lose
Episode: "All's Fair"

Topic: Game Theory with Unknown Types **Grade Level:** 9 - 12

Objective: Model a game with incomplete information

Time: 30 minutes

Introduction

In "All's Fair," Charlie attempts to use social image typology to analyze motivations and strategies to determine which person(s) would be most likely to commit a terrorist act. With social image typology, one can study why people make decisions that they do. Based on what they know about each suspect, Charlie and Don can assign a probability of committing a crime to each of them

The use of social image typology stems from game theory dating back to Auguste Cournot in 1838. Game theory is a branch of mathematics that studies strategic situations in which players choose actions to maximize their returns. Even though economists worked on developing methods for studying strategic interaction, breakthroughs were primarily made with the work of mathematician John Nash ("A Beautiful Mind") and later by John Harsanyi (1920 - 2000). Whereas Nash based his work on the assumption that the players know each other's preferences or "types," Harsanyi considered games where there is incomplete information on different players of the games. So important was Harsanyi's work that he was awarded a Nobel Prize for Economics in 1994 for his work in developing the analysis of games of incomplete information.

In a game with incomplete information, a Bayesian game, the question becomes how we deal with the uncertainty players face in not knowing all the motivations and strategies of their opponents.

This is similar to the situation in which Charlie and Don find themselves during a search for a terrorist. Harsanyi believed that players have certain preferences and that there was a subjective probability that could be assigned to their preferences. In Charlie and Don's case, the terrorists could be interested only in money, or they could be dedicated to a cause.

From a game theory standpoint, Harsanyi turned the game from one with incomplete information to one with imperfect or imprecise information (as students will see in question 4 in the student activity).

As an example, consider the matching pennies game in which two players show either heads or tails. If both coins match, then player A wins. If the coins do not match, player B wins. In a game where both players randomly decide whether to show heads or tails, there is no advantage to either player as shown in the following tree diagram.

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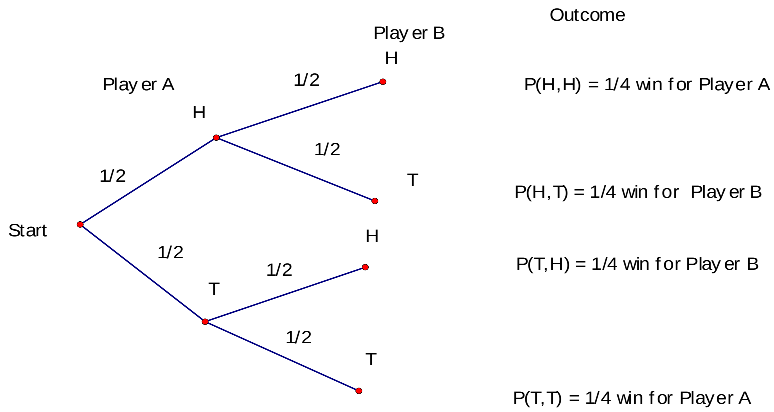

Discuss with Students

1. If the probability of an event occurring is x, what is the probability of the event not occurring?

2. If you play the game as described in the example above, with what probability does player A assume that she will win?

Suppose player B thinks that player A has a tendency to play heads 60% of the time and tails 40% of the time. Also suppose that player A has no idea how player B will play, or that player A assumes that player B plays at random.

3. With what probability should player B play heads in order to win?

4. To calculate the *mathematical expectation* of a game, first find the product of the probability of winning and the amount of the prize. If there is more than one way to win, sum each of the products.

Then subtract this total from amount paid in order to play. For example, the mathematical expectation for winning a \$1,000,000 lottery with a probability of winning to be 1/2,000,000 costing \$1.00 to play is: $(\$1{,}000{,}000)\left(\frac{1}{2{,}000{,}000}\right) - \$1.00 = -\$0.50.$

This means that the player should expect to lose \$0.50.

What if the amount of the prize is now doubled to \$2,000,000? How does that change the mathematical expectation of winning?

Discuss with Students Answers:
1. $1 - x$ **2.** *From player A's standpoint and symmetry in the probability tree diagram, the expectation is that winning will occur about 50% of the time* **3.** *Player B should choose heads less than 50% of the time to expect to win in the long run.* **4.** *The player expects to break even; that is the expected winnings equals the cost to play.*

Student Page Answers: 1a. P(H, H) = 0.6x; P(H, T) = 0.6(1 – x); P(T, H) = 0.4x; P(T, T) = 0.4(1 – x) **1b.** Each has probability 0.5 of winning. **1c.** $x < 1/2$ **2a.** 30,000(1-x) + 0(x) **2b.** $x < 2/3$. With a probability less than 2/3, the terrorist expects to earn more than \$10,000 If the act were to be repeated many times with the same conditions. **3.** Stop and take the \$10,000. He has no control over the action of suspect 2 and would wind up \$0 because suspect 2 will choose the \$40,000 payoff. **4a.** 10,000q +10,000(1 – q), or \$10,000 **4b.** (1 - q)0 + 15,000q >10,000, or 15,000q>10,000, or $q > 2/3$ **4c.** Because 3/5 is less than 2/3, the FBI knows that suspect 1's best strategy is to stop and collect \$10,000. If suspect 1 decides to go on, his mathematical expectation is (3/5)(\$15,000) or \$9,000. This is less than if he stopped. Knowing this, the FBI should concentrate on suspect 2.

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Name: ________________________________ Date: ______________

NUMB3RS Activity: Guess My Type or Lose

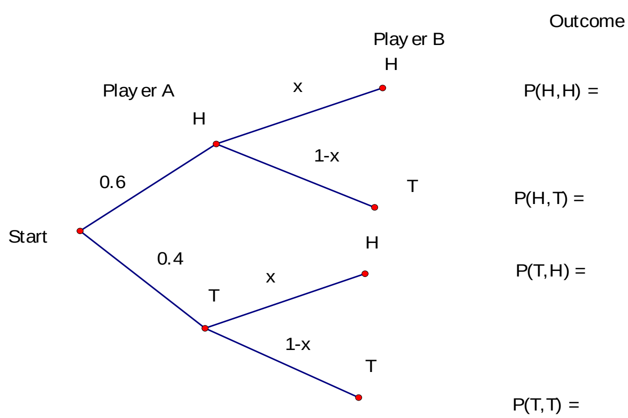
[This activity is based primarily on an article by Kevin McCabe, which is referenced in the resources.]

In “All’s Fair,” Charlie helps Don find a mathematical model with which Don can choose proper suspects with some degree of confidence based on the suspects’ actions. Charlie talks about using game theory to analyze how this might be done. Game theory is a branch of mathematics that studies strategic situations in which players choose actions to maximize their returns.

Charlie considers the needed model to be similar to a game in which incomplete or missing information about the suspects can be turned into one with imperfect or imprecise information. With imperfect or imprecise information, the game can be analyzed, and an examination of the game may lead to realistic decisions. The method used to assign the probabilities for the incomplete information is one of letting nature decide, or assigning random probabilities.

1. The tree diagram below shows two players who are matching pennies. If both coins match, player A wins. Otherwise player B wins. Player B thinks that Player A is showing Heads about 60% of the time and Tails 40% of the time. Based on this knowledge Player B wants to decide with what probability to play Heads or Tails to have the best chance of winning.

a. Find algebraic expressions for the indicated outcome probabilities *P*(H, H), *P*(H, T), *P*(T, H), and *P*(H, H). Let *x* represent the probability that Player B chooses Heads.



b. Suppose that Player B chooses to play H or T at random, show that each of the players expect to win half of the time.

c. Use the algebraic expressions found in part **a** to find a value for x so that Player B can expect to win more times than he or she expects to lose.

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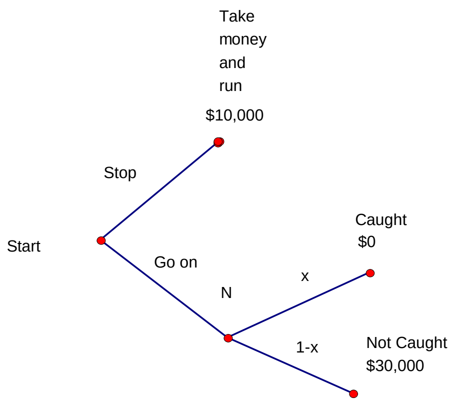
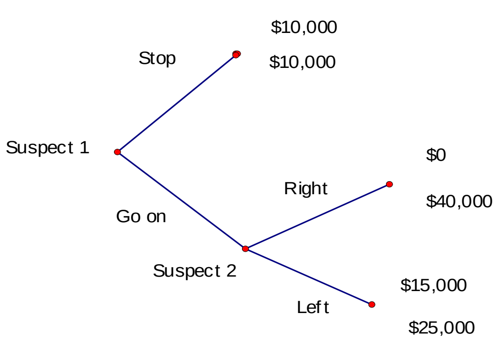
2. The tree diagram below depicts a suspect who has a decision to make: follow beliefs to perform a terrorist act, or simply accept the $10,000 that has been already been paid for a previous act and run. At the next branch of the tree labeled N, Charlie and Don are analyzing the needed probability *x* that will help them decide whether the terrorist will go ahead with the act of terrorism or will simply stop and gain no more money.



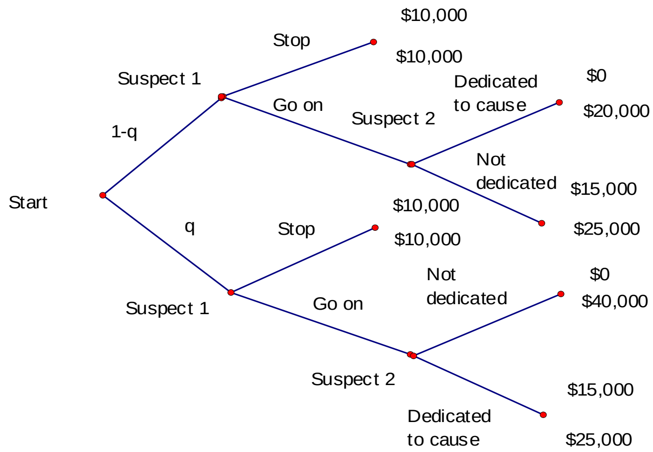
a. The suspect must decide whether to stop or go on without knowing the probability of getting caught, *x*. If the suspect goes on, what is the algebraic expression showing his expected payoff? [Hint: Find the product of the probabilities and payoff for getting caught and not getting caught, and add the two products.]

b. Suppose the suspect will perform a second act of terrorism if the expected payoff for the second act is greater than the $10,000 paid for the first act. What would the value of x need to be for the suspect's payoff for the second act to be greater than the money in hand already? Explain your reasoning.

3. The FBI has set up the following tree diagram below depicting the choices for two suspects. The FBI and each of the suspects have complete information about both suspects and their desire to make the most money possible. Suspect 1 can decide that they split $20,000, receiving $10,000 each for a potential job or they can go on. If suspect 1 decides that they go on, suspect 2 makes the rest of the decisions simply labeled right and left in the tree diagram. Suspect 2 could do the job alone and receive $40,000 or he could decide that they do the job together with suspect 1 receiving $15,000 and suspect 2 receiving $25,000. In the tree diagram, the money that suspect 1 expects to receive is the top dollar figure and the amount that suspect 2 expects to receive is the bottom dollar figure on each branch. If both suspects are interested in making the most money, what action does the FBI believe suspect 1 will take?



4. Now suppose that the situation is changed so that suspect 1 has no idea what suspect 2 will choose to do. For example, suspect 2 may simply be interested in the money, or may be more interested in the cause and less so the money. The new tree diagram shows when suspect 1 **believes** that there is a probability *q* for suspect 2 being dedicated to the cause. Regardless of what suspect 1 believes, suspect 2 could be dedicated or not and will make an independent choice on the terrorist act. The nodes of the tree marked Suspect 1 (or Suspect 2) show when that suspect is making a decision.



a. Write an algebraic expression for the amount of money suspect 1 receives if he decides to stop regardless of his beliefs about suspect 2.

b. If suspect 2 is dedicated to the cause, he is more likely to take the least amount of money possible. Thus, both the FBI and suspect 1 believe that suspect 2 is likely to make the choice of taking $20,000 (topmost branch of end section) or taking $25,000 (bottommost branch of end section). If suspect 1 is more interested in the money, write an algebraic inequality that will help him decide whether to stop or go on.
[Hint: If suspect 1 stops, he gets $10,000. If he goes on, his winnings will be based on suspect 2's actions.]

c. If the FBI has all of the information in part **a** and estimates the probability (*q)* of suspect 2 being dedicated to the cause to be 3/5, explain whether they should concentrate on suspect 2?

The goal of this activity is to give your students a short and simple snapshot into a very extensive math topic. TI and NCTM encourage you and your students to learn more about this topic using the extensions provided below and through your own independent research.

Extensions

- Explain all of the possible outcomes and decisions that could be made from the tree diagram in problem 4 of the student activity.

- Investigate the Rock-Paper-Scissors variation in the article by Kevin McCabe on page 193, of "Game Theory in Economics," found at the following Web site: **http://www.neuroeconomics.net/pdf/materials/454.pdf**

  How does this compare to the pure-strategy game of Rock, Paper, Scissors?

Other Resources

- An obituary of John Harsanyi can be found at the following Web site: **http://www.jewishvirtuallibrary.org/jsource/biography/harsanyi.html**

- McCabe, Kevin A. "Game Theory in Economics," found at the following Web site: **http://www.neuroeconomics.net/pdf/materials/454.pdf**

- This Web site contains an article "Bayesian Games" by Levent Koçkesen. **http://www.columbia.edu/~lk290/teaching/uggame/lecture/ugbayes.pdf**

- This Web site contains the article "Bayesian Games" by James Peck, University of Pennsylvania. **http://www.econ.ohio-state.edu/jpeck/Econ805/gametheory4.pdf**

- For a more detailed look at Bayesian filtering, see the activity **Filtering Suspects**, which accompanies the *NUMB3RS* episode "Judgment Call." This activity can be downloaded free from the Web site below:

  http://www.cbs.com/primetime/numb3rs/ti/activities/Ep202_Act1_JudgmentDay_final_9-15-05.pdf

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