

Calculus and the TI-89

By John F. Mahoney

The TI-89's symbolic capacities make it ideal for teachers and students of calculus. Here are some examples of how both can use the calculator to understand calculus better.

A. Introduction

The calculator's limit computation capabilities enable users to calculate derivatives using the definition of derivative:

$$\lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} = -\sin(x)$$

When the calculator is "asked" for the derivative of a quotient, it doesn't use the standard form:

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \cdot \frac{d}{dx}(f(x)) - f(x) \cdot \frac{d}{dx}(g(x))}{(g(x))^2}$$

To get the standard form, consider using the common denominator function:

$$\text{comDenom} \left(\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) \right) = \frac{g(x) \cdot \frac{d}{dx}(f(x)) - f(x) \cdot \frac{d}{dx}(g(x))}{(g(x))^2}$$

Here's a way to illustrate the chain rule:

$$\frac{d}{dx}(\sin(3x^2)) = 6x \cdot \cos(3x^2)$$

B. Implicit Functions

The TI-89 can graph implicit functions. Consider the curve $x^2 + xy + y^2 = 27$ which was the subject of a 1994 Advanced Placement problem. To graph the curve on the TI-89, change the Graph Mode to 3D, press $Y=$ and enter the curve as $z1 = x^2 + x \cdot y + y^2 - 27$.

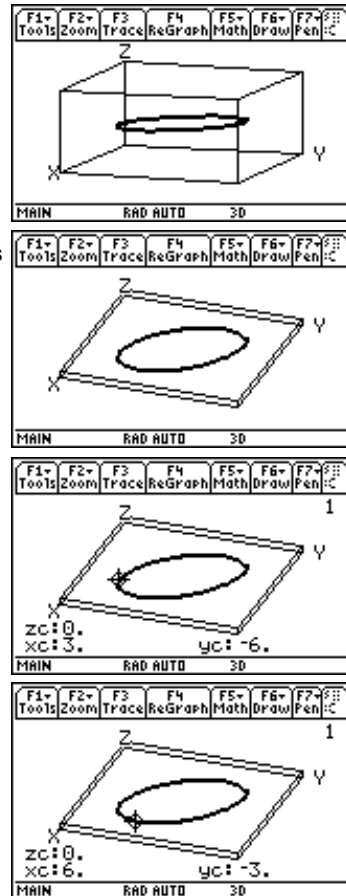
Press **F1** Tools, then **9** for Format, and set Coordinates to **Rect**, Axes to **Box**, Labels **On**, and Style to **Implicit Plot**.

Then press **Enter** to save these settings. You can either press **Zoom 6**: Std and wait about 90 seconds for the graph to appear, or you can choose your own settings.

In the top figure, the y-axis is horizontal, the z-axis is vertical, and the x-axis is coming out at you in standard form. The graph can be rotated and animated by using the arrow keys. I rotated the graph to look "down" the z-axis:

Here, the point (3, -6) is a point where the tangent line to the curve is parallel to the x-axis:

Whereas the point (6, -3) is a point where the tangent line to the curve is parallel to the y-axis:



How did we arrive at those two points? We can use implicit differentiation to do so. In order for the calculator to do implicit differentiation, one must first "tell" it that y is a function of x. Thus the equation of the curve is typed in with $y(x)$ in place of y . Then the expression is differentiated with respect to x .

$$\frac{d}{dx}(x^2 + x \cdot y(x) + (y(x))^2 = 27) = 2x + y(x) + 2y(x) \cdot \frac{d}{dx}(y(x)) = 0$$

One can solve for dy/dx by first subtracting $y(x)$ and $2x$ from both sides of the equation as follows:

And then dividing both sides by $(2 \cdot y(x) + x)$.

$$\frac{d}{dx}(y(x)) = \frac{-(y(x) + 2x)}{2y(x) + x}$$

Here, one can see that the tangent line to the curve will be horizontal when the numerator = 0, or when $y = -2x$. The tangent line to the curve will be vertical when the denominator = 0, or when $2y = -x$. We can solve the equation of the original curve when $y = -2x$ for x and likewise solve it when $y = x / -2$ to find the points where the tangents are vertical:

We can also transform the original equation by rotating through an angle of $\pi/4$ by making the substitution $x = (a - b) / \sqrt{2}$ and $y = (a + b) / \sqrt{2}$ as follows:

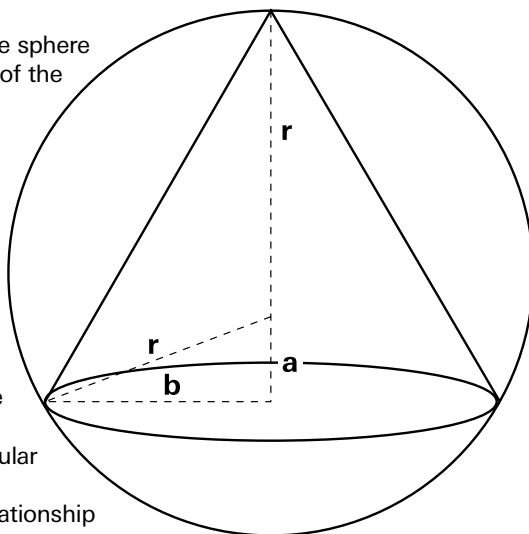
Finally, this equation can be put into the standard form of an ellipse as follows:

C: Maximum-Minimum

A typical hard maximum-minimum problem in calculus involves finding the volume of the largest right circular cone inscribed in a sphere of radius r . What is hard, for students, about this problem is both the set-up and the algebra involved in finding derivatives and their zeros. The TI-89 can't help with the set-up, but it can certainly help with the rest of the problem. This will allow students (and their teachers) to concentrate on exploring ways to set up problems — rather than allowing themselves to always get bogged down in the algebra. As a bonus, teachers can use this tool to quickly check the consistency of work of students who set up problems incorrectly.

Consider the following diagram showing a right circular cone inscribed in a sphere:

The radius of the sphere is r . The radius of the cone is b . The height of the cone is $r+a$ where a is the distance from the center of the base of the cone to the center of the sphere. Clearly, because we are dealing with a right circular cone, there is a Pythagorean relationship between a , b , and r .



The volume of the cone is $V = \frac{1}{3} \pi \cdot b^2 (r+a)$ where $a^2 + b^2 = r^2$.

Here, we first solve $a^2 + b^2 = r^2$ for a . The solver gives both a positive and a negative value for a . We then substitute the positive value for a (in terms of b and r) into our expression for the volume of the cone using the "with operator" — the vertical bar. This gives us an expression for the volume of the cone in terms of a single independent variable, b , and r — the radius of the sphere.

Now we differentiate our expression for the volume of the cone with respect to b :

We find the three zeros of the derivative:

We then find the second derivative with respect to b — note the $(b,2)$ in the command line. [The second derivative is quite a complicated expression!]

Now we evaluate the second derivative (above) at each of the critical points. It becomes obvious that the last critical point, $b = \frac{2\sqrt{2}}{3} r$ (the only positive one), gives a negative second derivative and thus it will be the value of b which produces the cone of maximum volume.

We then go back and evaluate our expression for the volume of the cone at the critical point

$b = \frac{2\sqrt{2}}{3} r$ to find the actual volume of the cone in terms of r . We can simplify our answer by finding the value of the volume when $r = 1$ in order to make the conclusion that the maximum volume of the cone is $\frac{32\pi}{81} r^3$ which turns out to be $\frac{8}{27}$ of the volume of the sphere!