EXPLORATIONS™

Activity 15

Approximating Integrals with Riemann Sums

Objectives

- Calculate Riemann sums
- Demonstrate when Riemann sums will over-approximate or under-approximate a definite integral
- Observe the convergence of Riemann sums as the number of subintervals gets larger

Materials

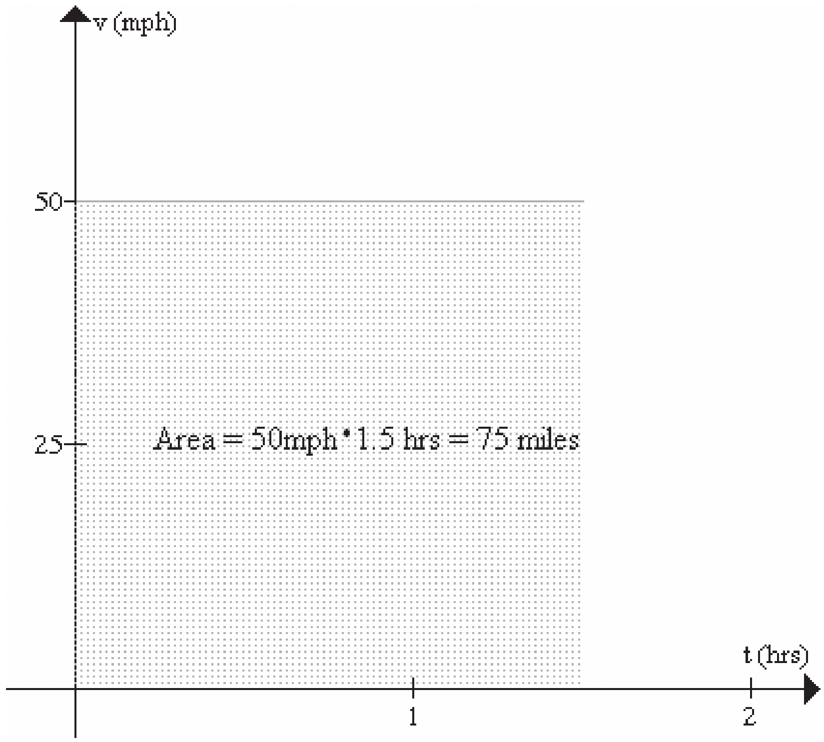
- TI-84 Plus / TI-83 Plus
- NUMINT program

Introduction

To measure the amount a quantity has changed, a definite integral can be used. In order to use the definite integral, you need to know the *rate* at which the quantity is changing. For example, if you know the rate at which your distance is changing with respect to time (that is, velocity) during a trip, then you can use the integral to calculate the distance you have traveled. If you know the rate at which the world consumed fresh water during a particular interval of time, then you can figure out how much fresh water was consumed during that particular time interval. In this activity, you will see how to approximate the value of definite integrals.

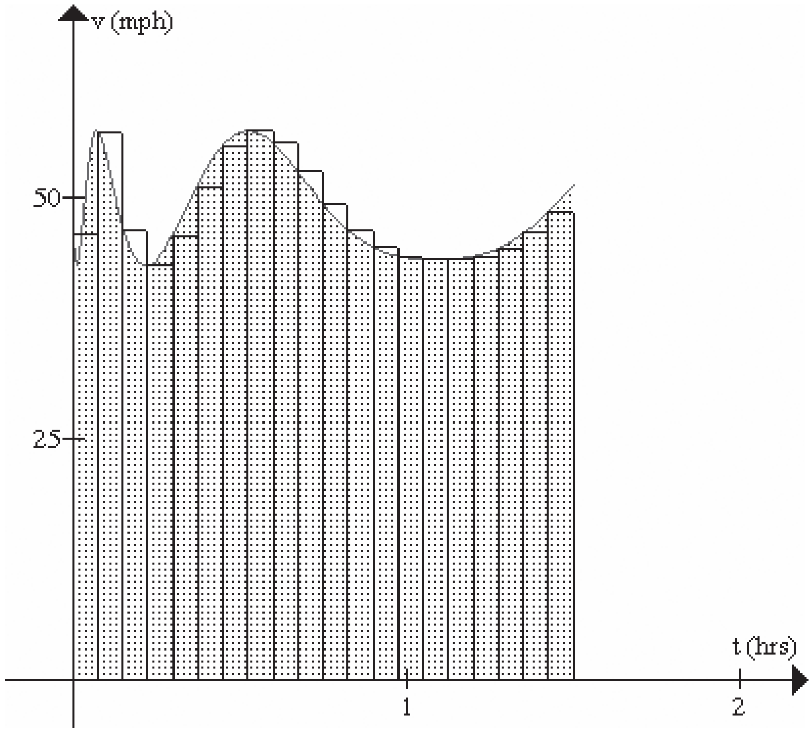
Exploration

Suppose that you drove down a highway at 50 miles per hour on cruise control. Over a period of 1.5 hours, you would have traveled 50 miles/hour(1.5 hours) = 75 miles during that time interval. The distance that you traveled is equal to the area under the graph of velocity versus time from $t = 0$ to $t = 1.5$ hours. It is not necessary to use an integral to calculate distance when the velocity is constant.



Assume that you are not on cruise control. You are driving on a winding country road, and your velocity is not constant. You must assume that your velocity is constant over a number of short time intervals. Multiplying the velocity by the short elapsed time will tell you a small amount of distance traveled.

Repeating the multiplication operation over many small intervals of time and adding up all the small distances will get the total distance traveled. See the figure on the next page.



Notice that the area under the curve at each subinterval is approximated by the area of the rectangle at each subinterval. All of the rectangular areas are added together, and the sum is used to approximate the exact area, which is the value of the integral.

In this activity, you will work with a rate that is given by the formula

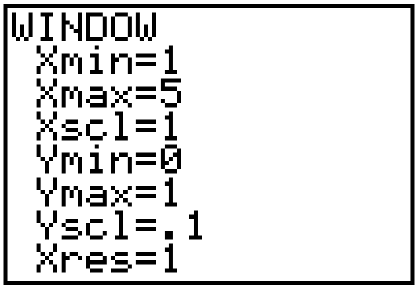
$$f(x) = \frac{1}{x^2}$$

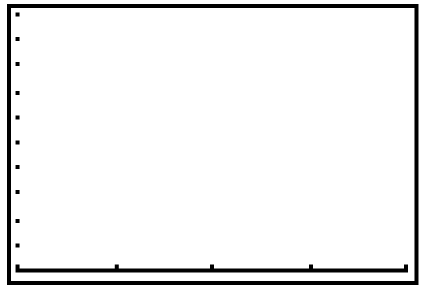
In more practical examples, you might be given rates in a table of values with no defining formula. Here, you will approximate

$$\int_1^5 \frac{1}{x^2}\,dx$$

The first step in approximating an integral is to subdivide your interval. In this example, work with the interval from $x = 1$ to $x = 5$.

1. Input **Y1** as **1/X^2** in the **Y=** editor, and graph it in the viewing window shown. Copy the graph on the axes shown. Note that the left boundary of the graph is $x = 1$, not the y-axis.





First, use 2 subintervals, [1, 3] and [3, 5]. Evaluate your rate function, **Y1** $= \frac{1}{x^2}$, at the left-hand endpoint of each of these intervals, $x = 1$ and $x = 3$.

On the **TBLSET Menu**, set **TblStart** to 1 and **ΔTbl** to 2.

2. View the table and record the values.

X	**Y1(X)**
1	
3	

3. On the graph you drew for Question **1**, draw rectangles with width 2 and heights given by the values of **Y1** from the table.

4. Calculate the area of each of your rectangles by multiplying each value of **Y1** from the table by the width of the rectangle, 2. Fill in the table.

Rectangle from	**Area**
1 to 3	
3 to 5	

5. What is the sum of the areas of the two rectangles?

This sum is called the left-hand Riemann sum with 2 subintervals for **Y1** on the interval [1, 5]. Look at the rectangles you drew on your graph.

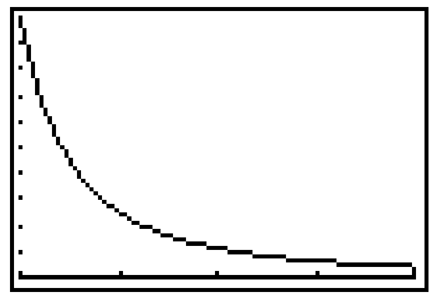
6. Is the value of your left-hand sum smaller or larger than the exact area under the graph of **Y1**? Explain.

To improve the approximation of the area, you need to include more subintervals so that the width of each rectangle is smaller. Use 8 rectangles, each having width 0.5.

7. On the **TBLSET Menu**, change **ΔTbl** to **0.5**. Record the values of **Y1** in the table shown.

X	Y1(X)
1	
1.5	
2	
2.5	
3	
3.5	
4	
4.5	

8. On the graph shown, draw the 8 rectangles with width 0.5 and heights given by the values of **Y1** from the table.



9. Calculate the area of each of your rectangles by multiplying each value of **Y1** from the table by the width of the rectangle.

Note: *A quick way to do this is to define* **Y2** = *0.5****Y1** *and view the table.*

Fill in the table shown.

Rectangle from	Area
1 to 1.5	
1.5 to 2	
2 to 2.5	
2.5 to 3	
3 to 3.5	
3.5 to 4	
4 to 4.5	
4.5 to 5	

10. What is the sum of the areas of the eight rectangles?

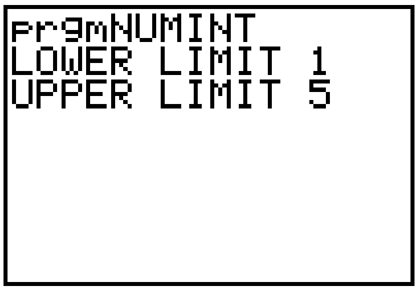
11. Is the sum larger or smaller than the exact area under the curve?

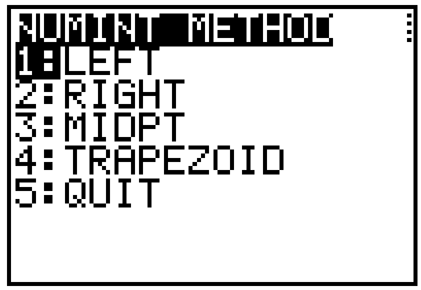
12. Is the sum larger or smaller than the sum you calculated using only two rectangles?

Draw a picture that supports your answer.

Generally, using more rectangles in the sum gives a better approximation. However, these calculations soon become tedious. A graphing handheld program can automate the process.

From the **PRGM Menu**, select **NUMINT** and press [ENTER] to run the program. The graphing handheld draws the graph of **Y1** in the current viewing window you have set and prompts you for a lower limit. Enter the left endpoint of your interval, **1**. It then prompts for the upper limit. Enter the right endpoint of your interval, **5**. It then prompts you to select from a menu of approximation methods. Select **1:LEFT**. See the screenshots below.





You are then prompted for the number of subintervals. This is the number of rectangles you want to use. Enter **8** to check your work from Questions **8–12**. You will see the rectangles drawn on the graph, after which the program pauses. Press [ENTER]. The total area is displayed (the sum of the areas of the 8 rectangles), and you are prompted again for the number of subintervals. Use the program to calculate several left-hand Riemann sums. Note that when you get to 128 rectangles or so, the graphing handheld takes a few moments to draw the rectangles and calculate the area.

Note: *In the upper right corner of the screen, a train of several points appears to be moving up the screen. When the train of points turns into a dotted line, the calculations are complete. You may then press* [ENTER] *to view the area and be prompted for the next entry.*

13. Complete the table, recording the Riemann sums accurate to the ten-thousandths place. To get back to the **NUMINT METHOD Menu**, enter **0** for the number of subintervals. You can then choose **2:RIGHT** to calculate the right-hand Riemann sums.

Number of Subintervals	Left-Hand Riemann Sum	Right-Hand Riemann Sum
8		
16		
32		
64		
128		
256		
512		
1024		

14. Do the values of the left-hand Riemann sums approach a limiting value as the number of subintervals gets larger? If so, what is that value?

15. Are the right-hand Riemann sums larger or smaller than the exact area under the curve?

16. Explain why the right-hand Riemann sums increase as the number of rectangles increases.

17. Do the values of the right-hand Riemann sums approach a limiting value as the number of subintervals gets larger? If so, what is that value?