

Metallic Numbers

Teacher Notes & Answers

7 8 9 10 11 12



Introduction

The famous Fibonacci sequence 1, 1, 2, 3, 5, 8 ... involves the recursive sequence: $t_{n+2} = t_n + t_{n+1}$.

The ratio between consecutive terms for the Fibonacci sequence as $n \rightarrow \infty$ is known as the Golden Ratio.

$$\text{Golden Ratio: } \lim_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} = \phi$$

In this investigation you will explore a small variation on the Fibonacci sequence: $t_{n+2} = t_n + at_{n+1}$ where a is a natural number. In this investigation these variants on the Fibonacci sequence will be referred to as “Levels”, for example Fibonacci Level 2” means that $a = 2$ in the recursive definition above. The original Fibonacci sequence is therefore Fibonacci Level 1 with $a = 1$.

Fibonacci Level 2: In search of the Silver Ratio

This sequence starts as: 1, 1, 3, 7, 17, 41, 99 ...

Each successive term is equal to “the previous two terms plus another helping of the previous term.” This can be expressed more succinctly using mathematical notation as:

$$t_{n+2} = t_n + 2t_{n+1}$$

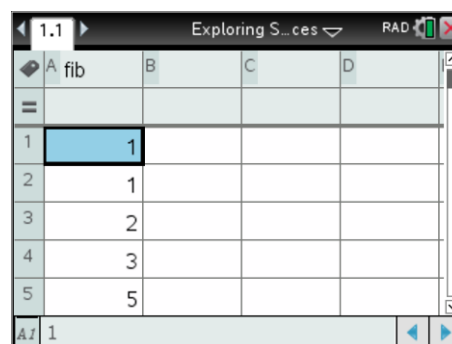
The first two numbers can still be set as 1 and 1.

Use either a recursive formula or an appropriate sequence command to generate the first 50 terms of the Fibonacci Level 2 sequence.

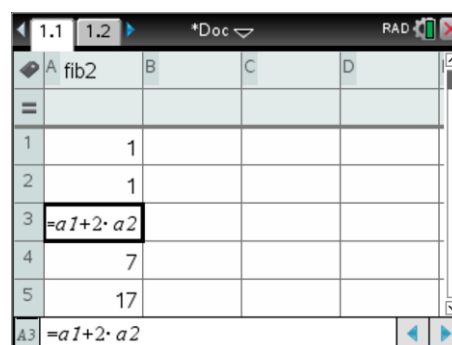
Call the list: FIB2

Insert a Calculator Application in preparation for your exploration.

An appropriate formula for the Fib2 list is shown opposite:



A	B	C	D
fib			
=			
1	1		
2	1		
3	3		
4	7		
5	17		
A1	1		



A	B	C	D
fib2			
=			
1	1		
2	1		
3	=a1+2*a2		
4	7		
5	17		
A3	=a1+2*a2		

Question: 1.

Explore the ratio between consecutive Fibonacci Level 2 terms, this is called the ‘Silver’ ratio.

Note: Any term in the sequence can be recalled by typing: FIB2[#] where # represents the term number.

Answers will vary with regards to exploration. Example $\text{FIB2}[11] / \text{FIB2}[10] \approx 2.4142139267767$. Students may already start to recognise this value as $\approx 1 + \sqrt{2}$. $\text{FIB2}[21] / \text{FIB2}[20] \approx 2.4142135623731$

Calculator Tip!

Insert a Notes application and a slider called 'n'. Set the minimum value of the slider to 1, the maximum to 99 with increments of '1'. In a maths box type:

$$\frac{\text{Fib2}[n+1]}{\text{Fib2}[n]}$$

Adjust the slider to see how the ratio between consecutive terms changes.

Question: 2.

Change the first two terms in the Fibonacci Level 2 sequence and check to see if this changes the long term value of the ratio between consecutive terms.

The common ratio still holds, (assuming both initial terms are non-zero) however the number of terms required to find the common ratio may change.

Question: 3.

Let x represent any term in the sequence and y the next term.

- a) Explain the two formulas below:

$$r_n = \frac{y}{x} \quad \text{and} \quad r_{n+1} = \frac{2y+x}{y}$$

The first equation represents the ratio between consecutive terms according to how x and y have been defined above. The second equation simply follows the recursive definition for the level 2 Fibonacci sequence.

- b) Assuming the ratio between consecutive terms is approximately equal as $n \rightarrow \infty$ determine the value of the ratio.

There are several ways this problem may be tackled. Initially students may assume that with two equations and three unknowns, it is not possible to determine a specific value for the ratio, in this case however it is possible. Furthermore, CAS functionality provides only a very limited advantage if any.

$xr_n = y$	(Eqn 1)	$\frac{y}{x} = \frac{2y+x}{y} \quad (r_{n+1} \approx r_n)$ $y^2 = 2xy + x^2$ $y^2 - 2xy - x^2 = 0$ $(y-x)^2 - 2x^2 = 0$ $(y-x-\sqrt{2}x)(y-x+\sqrt{2}x) = 0$ $y = (1 \mp \sqrt{2})x \quad (\text{Sub into Eqn 1})$ $r = 1 + \sqrt{2} \quad (r > 0)$
$r_{n+1} = \frac{2xr_n + x}{xr_n}$	(Sub. Eqn 1 into Eqn 2)	
$r_{n+1} = \frac{2r_n + 1}{r_n}$	($x \neq 0$)	
$r^2 - 2r - 1 = 0$	($r_{n+1} \approx r_n$)	
$(r-1-\sqrt{2})(r-1+\sqrt{2}) = 0$		
$r = 1 + \sqrt{2}$	$r > 0$	

Fibonacci Level 3: In search of the Bronze Ratio

The bronze ratio refers to the ratio between consecutive terms of the level 3 Fibonacci sequence. The general formula for the sequence $t_{n+2} = t_n + at_{n+1}$ therefore becomes: $t_{n+2} = t_n + 3t_{n+1}$

Question: 4.

Create a new list in the spreadsheet application called Fib3, generate the first 50 terms of the level 3 sequence and explore the ratio between consecutive terms as n increases.

Content generated on the calculator. Students should explore the ratio using a similar technique to that used in Questions 1 and 2.

With $t_n = 1$ & $t_{n+1} = 1$

FIB3[11]/FIB3[10] \approx 3.3027756406463

FIB3[21]/FIB3[20] \approx 3.302775637732

These two examples show that the ratio appears to once again be approaching a limit. Changing the initial terms doesn't make any difference to the eventual limit.

Question: 5.

Set up two formulas similar to those from Question 3 and hence determine the exact value for the bronze ratio.

$$r_n = \frac{y}{x} \quad (\text{Eqn 1}) \qquad r_{n+1} = \frac{3y+x}{y} \quad (\text{Eqn 2})$$

$$xr_n = y \qquad (\text{Eqn 1\#})$$

$$r_{n+1} = \frac{3xr_n + x}{xr_n} \qquad (\text{Sub. Eqn 1\# into Eqn 2})$$

$$r_{n+1} = \frac{3r_n + 1}{r_n} \qquad (x \neq 0)$$

$$r^2 - 3r - 1 = 0 \qquad (r_{n+1} \approx r_n)$$

$$\left(r - \frac{3}{2} - \frac{\sqrt{13}}{2}\right) \left(r - \frac{3}{2} + \frac{\sqrt{13}}{2}\right) = 0$$

$$r = \frac{3 + \sqrt{13}}{2} \qquad r > 0$$

Fibonacci Level n : The Metallic Ratios

The general term for the ratio between consecutive terms for $t_{n+2} = t_n + at_{n+1}$ is referred to as a Metallic ratio.

Question: 6.

Determine an expression for the general form of the Metallic ratios. Check your answer using $a = 1$, $a = 2$ and $a = 3$.

$$r_n = \frac{y}{x} \quad (\text{Eqn 1}) \qquad r_{n+1} = \frac{ay + x}{y} \quad (\text{Eqn 2})$$

$$xr_n = y \quad (\text{Eqn 1\#})$$

$$r_{n+1} = \frac{axr_n + x}{xr_n} \quad (\text{Sub. Eqn 1\# into Eqn 2})$$

$$r_{n+1} = \frac{ar_n + 1}{r_n} \quad (x \neq 0)$$

$$r^2 - ar - 1 = 0 \quad (r_{n+1} \approx r_n)$$

$$\left(r - \frac{a}{2}\right)^2 - \left(\frac{a^2 + 4}{4}\right) = 0$$

$$\left(r - \frac{a}{2} - \frac{\sqrt{a^2 + 4}}{2}\right) \left(r - \frac{a}{2} + \frac{\sqrt{a^2 + 4}}{2}\right) = 0$$

$$r = \frac{a + \sqrt{a^2 + 4}}{2} \quad r > 0$$

By Substitution for $a = 1$ the golden ratio: $r = \frac{1 + \sqrt{5}}{2}$ and the silver ratio $a = 2$, $r = \frac{2 + \sqrt{8}}{2} = 1 + \sqrt{2} \dots$

Question: 7.

For the golden ratio (ϕ) the following relationships hold:

$$\phi = \frac{1}{\phi} + 1$$

$$\phi^2 = \phi + 1$$

$$\phi^1 + \phi^2 = \phi^3$$

Do any of the above relationships hold for the silver or bronze ratio?

Students may use the specific ratios or go straight to the general case. Students should also note that all three of the above relationships are the same and therefore if the silver or bronze relationship hold for the first relationship it would therefore hold for the second and third.

$$r = \frac{a + \sqrt{a^2 + 4}}{2}$$

$$\phi^2 = \phi + 1$$

$$\left(\frac{a + \sqrt{a^2 + 4}}{2}\right)^2 = \frac{a + \sqrt{a^2 + 4}}{2} + 1$$

$$\frac{a\sqrt{a^2 + 4}}{2} + \frac{a^2}{2} + 1 = \frac{a + \sqrt{a^2 + 4}}{2} + 1$$

$$(a-1)\sqrt{a^2 + 4} + a^2 - a = 0$$

$$(a-1)\sqrt{a^2 + 4} + a(a-1) = 0$$

$$(a-1)(\sqrt{a^2 + 4} + a) = 0$$

$$a = 1 \quad \sqrt{a^2 + 4} + a > 0$$

The only valid solution for the relationships occurs when $a = 1$ (The Golden Ratio)

Question: 8.

Calculate the approximate value for each of the following and comment on your finding as the quantity of 'embedded' fractions increases.

a) $1 + \frac{1}{1+1}$

Answer = 1.5

b) $1 + \frac{1}{1 + \frac{1}{1+1}}$

Answer = 1.6*

c) $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1}}}$

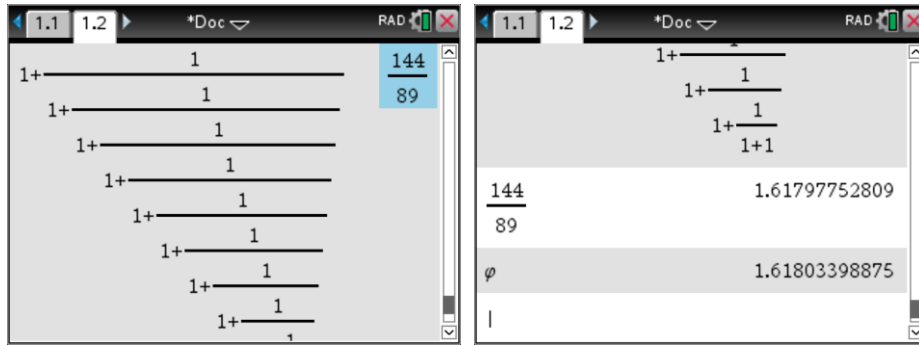
Answer = 1.6

d) $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1}}}}$

Answer = 1.625

e) $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1}}}}}$ (In this case add as many fractions as possible)

Answers will vary depending on how many embedded fractions students used:



Students should recognise that this recursive fraction is approaching the golden ratio. This can be shown algebraically:

$$\text{Let } x = 1 + \frac{1}{x}$$

To show how this relates to the above, consider $x_n = 1 + \frac{1}{x}$ substituting x_n into x .

The above substitution results in: $x = 1 + \frac{1}{1 + \frac{1}{x}}$ which can then be substituted again ... and again.

But $x = 1 + \frac{1}{x}$ can be expressed as $x^2 - x - 1 = 0$ the solution for which is Φ .

Question: 9.

Calculate the approximate value for the following 'embedded' fraction and comment on the result.

$$2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}}$$

Students should recognise the result as the silver ratio and hypothesise the general case for the metallic ratios:

$$a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \dots}}}}$$