Buffon’s Needle Extension

Answers

7 8 9 10 11 **12**

TI-Nspire Investigation Student 45 min

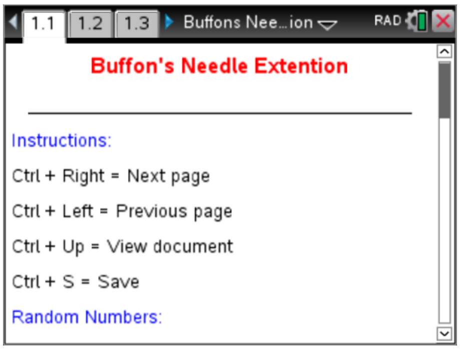
Problem

Your pen just rolled off the desk onto the timber floor, as you lean over to pick it up you notice that your pen doesn’t cross any of the joins in the floorboards. This brings back memories from your early childhood days of jumping over the cracks in the pavement. It was relatively easy to avoid the cracks in the pavement because your foot was so much smaller than the distance between the cracks. Your pen, on the other hand, seems to be exactly the same length as the distance between joins in the floor boards.

Curious, you pick up your pen and purposely drop it again. This time your pen crosses one of the joins. Not wanting to end on ‘bad luck’ you drop your pen once again, another hit. This is not good! You try again, this time it lands neatly between the cracks, a miss; good luck has returned. Distracted by these events you sit and ponder; “What are chances that when you drop your pen it will land over a crack?”

Simulating the Event

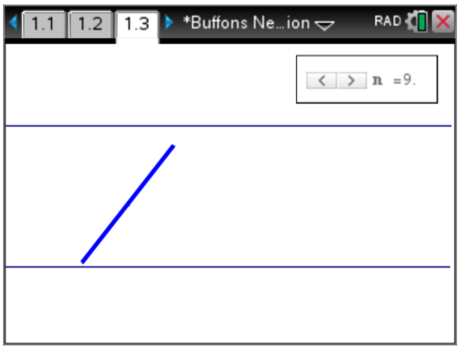
Open the TI-Nspire file: **Buffon**



Page 1.1 contains a set of instructions. These instructions include setting the random number generator.

Navigate to page 1.2 and enter the command: randseed followed by a four digit number. Make sure your four digit number is unique, such as the last four digits of your phone number, then press [**ENTER**].

Navigate to page 1.3.



Move the mouse over the slider and click to simulate a pen drop. The animation will highlight the line and display a HIT when the pen lands across a line. Try a few to make sure everything is working.

Once you have finished experimenting, change the value of *n* back to 1, then simulate and record 100 pen drops.

Question: 1.

Record the number of ‘hits’ out of 100 pen drops.

Answers will vary, typically between 50 and 70 hits.

Author: P. Fox

Question: 2.

Based on your experiment, what is the probability of your pen landing on a crack?

Answers will vary, but calculation should be: $\frac{\text{Number of hits}}{100}$

Question: 3.

Collect results from four of your friends. Combining this new data with your own, estimate the probability that your pen would land over a crack when dropped.

Answers will vary, but in order to establish a reasonable expectation of the probability a very large number of samples need to be conducted.

Comment / Opportunity:

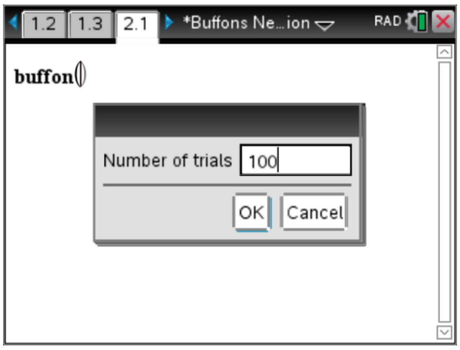
For schools with access to TI-Navigator this is an opportunity to introduce "Sampling Distributions". Send a Quick Poll to students using the LIST option. Student can use the program on page 2.1 to quickly generate multiple samples consisting of 100 trials. In the Quick Poll students can contribute multiple data points. If each student contributes 5 results, with 20+ students in the class this produces 100 samples of size 100, enough to produce a distribution that is approximately normal. A follow up quick poll can collect a data for sample sizes of 1000. Ask students to predict the shape of this distribution including reference to the mean and standard deviation.

For schools without a TI-Navigator system, an extra problem exists on page 4.1 with a program titled: Buffondist. This program allows the user to generate multiple samples of size 'n' and stores the data in a list called "data". This list can be copied and another data set with different sample sizes and the distributions compared in much the same way. This program will however take a long time to run on the handheld if 100 samples of size 100+ are generated.

Navigate to page 2.1.

A program to simulate the dropping of the pen will automatically check the results of 100's trials. To access the program press the [**VAR**] key and select: **Buffon**.

Press [**Enter**] to run the program and enter the number of trials.



Question: 4.

Run the program 10 times completing 1000 trials in each. Record the results for your 10 simulations.

a. Discuss the variation in your results with regards to the predictability of the *true* proportion or probability.
Answers will vary, however due to the larger sample size the variation in results (standard deviation) is much smaller. Results consistently fall in the mid 600's.

b. According to your data what is the best estimate for the probability your pen will land over a crack when dropped?
Answers will vary. The mean will be approximately 637.

Author: P.Fox

Question: 5.

Get results from 9 other friends for their average over 10,000 trials and record the results in a table.

a. Discuss the variation in the expected values for 10,000 trials.

The variation in results will be even smaller as n increases: $\frac{\sigma}{\sqrt{n}}$

b. Use all of these results to again estimate the probability of your pen landing on a crack.

Answers will vary, the mean will be approximately 6366.

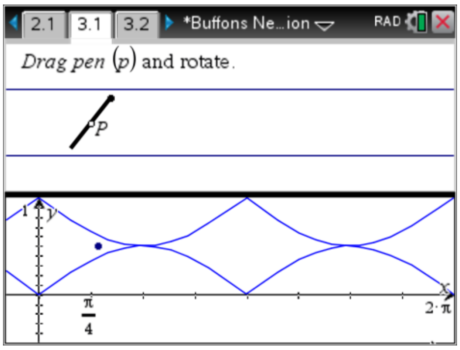
Question: 6.

Calculate the 'reciprocal' of your answer to Question 7, ("Flip it") and double the result. To what number is this similar? Is this a coincidence? – Discuss.

The result will be approximately 3.142. The theory behind this result is explored in the next section of this investigation. If students draw a diagram of the problem they may not be surprised by the result.

Consider the location of the centre of the pen as a random point between two parallel lines. This reduces the problem to a random point along a straight line which produces a uniform distribution. The pen can then be considered as a line rotating around its centre which produces a circle leading to the possibility relating to the involvement of π.

Calculating the Probability



Navigate to page 3.1 this page contains a diagram that can help calculate the probability that the pen will land over a crack.

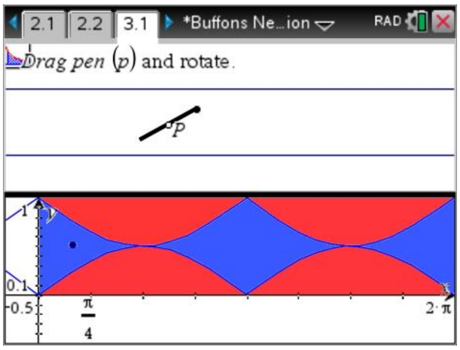
This time the pen can be manipulated manually. Drag point P so that the pen crosses the top line (crack) then the bottom.

While dragging point P notice how the representative point in the lower diagram moves.

Grab the tip of the pen to rotate. Observe what happens to the representative point in the lower diagram as you rotate the pen.

Question: 7.

Copy the graph (lower diagram). Using a red pen, shade the **region**(s) that correspond to when the pen crosses the crack. Use a blue pen to shade the **region**(s) that correspond to when the pen does not cross the crack. Explain how these regions relate to the probability.



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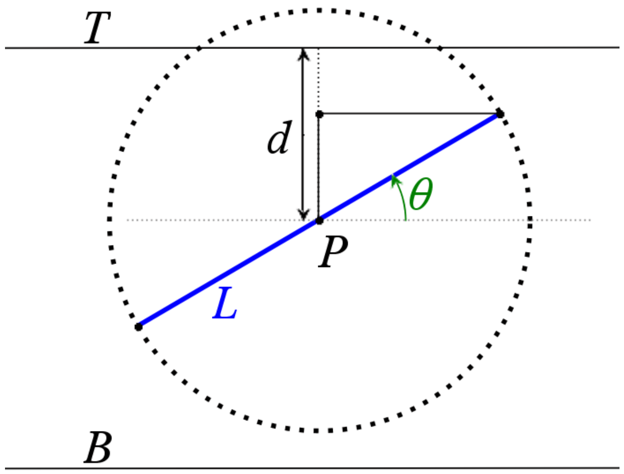
The regions are symmetrical so the a smaller section of the graph / diagram can be considered, for example $0 \le \theta \le \pi$ and $0 \le d \le \frac{1}{2}$.

The area under the graph (red) region represents favourable outcomes (crossing a join) and the blue region represents not crossing a join.



The diagram shown here illustrates the situation where the pen is exactly the same length (L) as the width of the floor boards.

The centre of a pen (P) that falls on the ground is located at a random point.

Distance (d) is measured from P to the floorboard join Line T.

Theta (θ) is the angle the pen makes with the floorboard join.

Question: 8.

Determine an inequality involving *d*, L and θ that will determine when the pen will cross the floorboard join line T.

d is the distance from P to the join, then if $d < \frac{L}{2}\sin(\theta)$ the pen would cross the join.

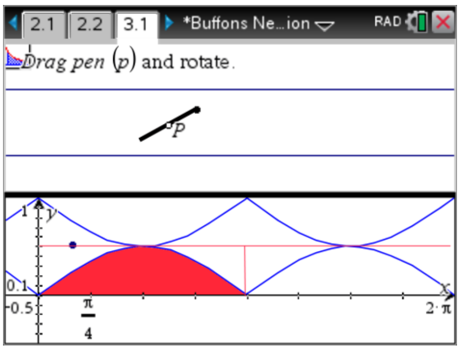
Question: 9.

Use the graph on page 3.1 and your answer to Question 8 to set up and calculate an appropriate integral for the pen crossing the floorboard join line

$$\frac{L}{2}\int_0^{\pi} \sin(\theta)d\theta = L$$

Question: 10.

Use your answers to Question 9 to determine the exact probability the pen will cross the floorboard join line and compare this with the probability estimates earlier in this activity.
(Assuming the pen is the same length as the distance between the lines.)



The region required has several lines of symmetry.
One example for the probability can be computed by:

$$\frac{\frac{L}{2}\int_0^{\pi} \sin(\theta)d\theta}{\frac{L}{2}\times\pi} = \frac{2}{\pi}$$

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