

Chapter 7

Differential Equations and Slope Fields

In Chapter 6, you used the **deSolve**(command to solve differential equations analytically. In this chapter, you will use the differential equation graphing mode of the TI-89 to solve differential equations graphically and numerically. You will start with a solution that should look familiar and then move to more difficult equations.

Before you begin this chapter, reset your TI-89 to the default settings by pressing **[2nd] [MEM] [F1] Reset**, then select **3:Default** **[ENTER] [ENTER]**.

Example 1: Slope fields and specific solutions

Solve this differential equation.

$$y' = \frac{x}{4}$$

Solution

The analytic solution is

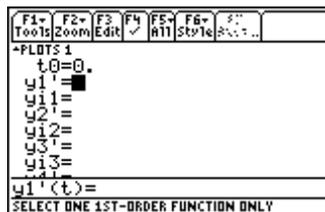
$$y = \frac{x^2}{8} + C$$

As you choose various values for C , you will obtain parabolas with the same shape that are shifted vertically from each other. You can graph this family of parabolas with the differential equation graphing mode on the TI-89.

1. Press **[MODE]** to display the MODE dialog box. Press **[▶]** and select **6:DIFF EQUATIONS**. Press **[ENTER]** to save the change and return to the Home screen.



2. Press \blacktriangleright [Y=] to display the Y= Editor. Clear any equations that are stored.



3. Enter the differential equation using t in place of x .

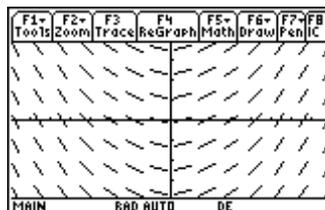
$T \div 4$ [ENTER]



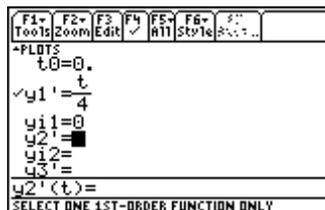
4. Press [F1] **Tools** and select **9:Format**. Change your format settings, as necessary, to match those shown at the right, and press [ENTER].



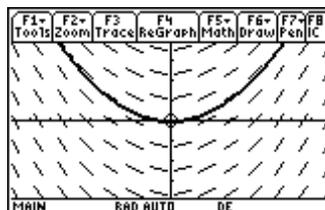
5. Graph the solution in a Zoom Decimal window by pressing [F2] **Zoom** and selecting **4:ZoomDec**. This graph is called a *slope field*. It suggests to the imagination a family of the parabolic solutions to the differential equation.



6. You can see specific members of this family of solutions by specifying initial conditions for the differential equation. If you want to see the specific graph corresponding to the initial condition $y(0) = 0$, return to the Y= Editor and enter **0** for $y1$.

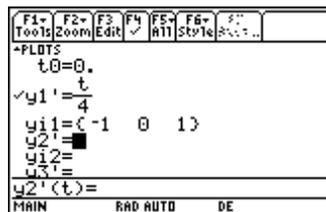


7. Press \blacktriangleright [GRAPH] to see the specific solution graphed with the slope field.

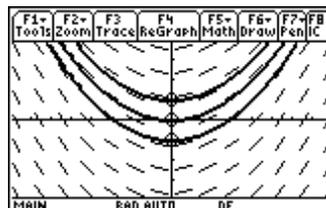


8. To see several members of the family of solutions corresponding to $y(0) = -1$, $y(0) = 0$, and $y(0) = 1$, enter a list of initial conditions in the Y= Editor.

2^{nd} [$\{$] [$\}$] 1 , 0 , 1 2^{nd} [$\}$] ENTER



9. Press \blacktriangleright [GRAPH]. This graph gives more insight into the slope field. If you start at any initial point in the slope field and move so that the line segments in the field are tangent to your path, you will trace out a specific solution to the differential equation.



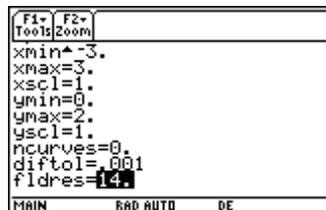
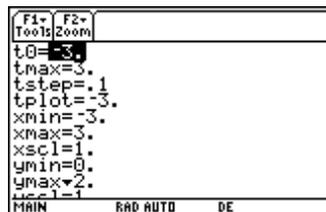
Example 2: Window variables and initial conditions

Graph the solution to

$$y' = e^{-x^2}, \quad y(-3) = 0$$

Solution

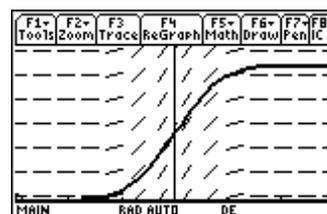
- In the Y= Editor, enter the differential equation as $y1'$ using t in place of x . Then enter the initial value of x in $t0$ and the initial value of y in $y1$.
- Select a $[-3,3] \times [0,2]$ viewing window. Display the Window Editor by pressing \blacktriangleright [WINDOW]. Enter the values for the Window variables as shown.



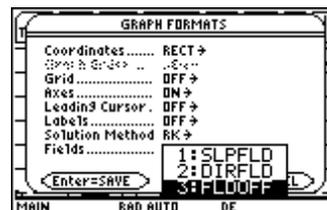
The Window variables shown in these examples have the following functions:

Variable	Function
tmin	The initial x -coordinate. This is the point at which the TI-89 will begin calculating the solution.
tmax	The point at which the TI-89 will stop calculating and graphing the solution (when the slope field is turned off)
tstep	The horizontal spacing between the points calculated in the solution
tplot	The point at which plotting begins (when slope field is turned off)
xmin	The left boundary of the viewing window
xmax	The right boundary of the viewing window
xscl	The spacing between the marks on the x -axis
ymin	The bottom boundary of the viewing window
ymax	The top boundary of the viewing window
yscl	The spacing between the points on the y -axis
ncurves	The number of specific curves drawn if no initial condition is specified
diftol	Affects the speed and accuracy of the solution when the format solution setting of RK is selected. (When Euler is selected, diftol is replaced with Estep .)
fldres	Affects the size of the segments shown in the slope field

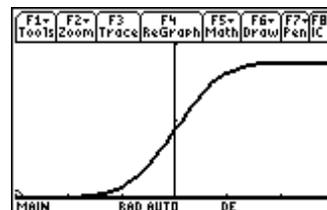
3. Press $\left[\text{GRAPH}\right]$ to see the solution.



4. Turn the slope field off by pressing $\left[\text{F1}\right]$ **Tools** and selecting **9:Format**. Set **Fields** = **3:FLDOFF**.



5. Press $\left[\text{ENTER}\right]$ to graph the solution. This graph is a scaled version of a normal probability density function. Probability density functions are used in probability and statistics.



Example 3: Tabular solutions to differential equations

The differential equation that describes the decay of carbon-14 is

$$\frac{dy}{dt} = -\frac{\ln 2}{5700}y, \quad y(0) = 1$$

where y is the proportion of radioactive nuclei present in a dead organism to radioactive nuclei present in the organism while it lived and t is elapsed time in years since the organism died.

- (a) Find the proportion of radioactive carbon-14 in a sample after 1000, 2000, 3000, 4000, and 5000 years.
- (b) How old is a sample in which 20% of the carbon-14 nuclei have decayed?

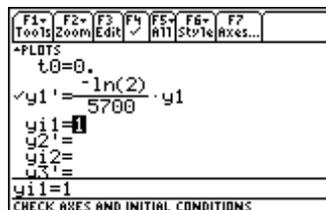
Solution

1. Make sure **Graph** mode is set to **DIFF EQUATIONS**.
2. In the Y= Editor, enter the differential equation along with the initial conditions.

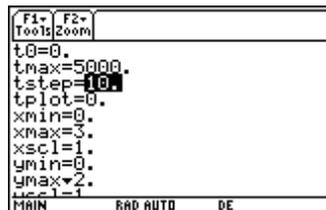
$$y1' = -\ln(2)/5700 * y1$$

$$t0 = 0$$

$$y11 = 1$$



3. In the Window Editor, set **tmax= 5000**, **tstep=10**, and **xmin=0**. All other values are the same as in Example 2.



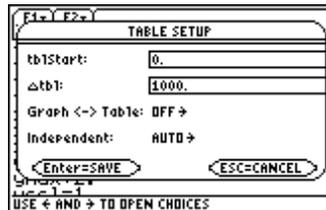
4. Press \blacklozenge [TblSet] and set the table parameters:

$$tblstart = 0$$

$$\Delta tbl = 1000$$

$$\text{Graph} \leftrightarrow \text{Table} = \text{OFF}$$

$$\text{Independent} = \text{AUTO}$$



5. When you have these settings, press [ENTER] twice to save them. Then press \blacklozenge [TABLE] to see the table of values for the solution to the differential equation. Scroll down to see the value for $t=5000$. The $y1$ column shows the proportion of radioactive carbon-14 after 1000, 2000, 3000, 4000, and 5000 years.

t	y1		
1000.	.8855		
2000.	.7841		
3000.	.69431		
4000.	.61481		
5000.	.54441		

t=5000.

6. You can see from the table that the answer to part (b) is somewhere between 1000 and 2000 years. Press \blacklozenge [TblSet] to display the TABLE SETUP dialog box and start the table at 1000 years (**tblstart**) with 100-year increments (**Δ tbl**).



7. Press ENTER twice to save the changes and see the new table. Scroll down the table until the second column is close to 0.80.

The answer is somewhere between 1800 and 1900 years.

t	y1				
1500.	.83326				
1600.	.82319				
1700.	.81324				
1800.	.80341				
1900.	.79369				

t=1900.
MAIN RAD AUTO DE

8. Return to the TABLE SETUP dialog box and start at 1800 with 10-year increments. This method of expanding the table in the area of interest is called *tabular zoom*.

It appears the sample is about 1830 years old.

The tables in this example were calculated using a solution method known as the *Runge-Kutta* (RK) method. You can select either the Runge-Kutta method or Euler's solution method in the for GRAPH FORMATS dialog box. The Runge-Kutta method is generally more accurate but, Euler's method is often faster. Consult a calculus book to see the details involved in these methods.

t	y1				
1800.	.80341				
1810.	.80243				
1820.	.80145				
1830.	.80048				
1840.	.79951				

t=1800.
MAIN RAD AUTO DE

9. Change to Euler's solution method by pressing \blacklozenge [Y=] F1 Tools and selecting **9:Format**. Set **Solution Method = EULER**.



10. Press ENTER to save the **Euler** setting. Then press \blacklozenge [TABLE] to see the table calculated with Euler's method. Notice the differences between this table and the table generated with the RK method in step 8.

t	y1				
1800.	.8033				
1810.	.80233				
1820.	.80135				
1830.	.80038				
1840.	.7994				

t=1800.
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Example 4: Higher-order differential equations

In this example, you will solve a second-order differential equation. Since the TI-89 only accepts first-order differential equations, you will need to use a set of substitutions to solve higher-order equations. These substitutions are:

$$y_1' = y_2$$

$$y_1'' = y_2'$$

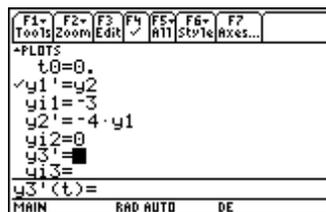
When you make these substitutions, you can enter the second-order differential equation in $y_2' =$. Solve

$$y'' = -4y, \quad y(0) = -3, \quad y'(0) = 0$$

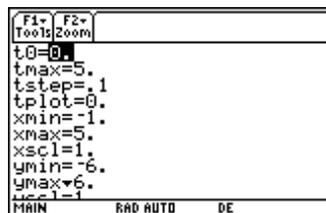
Solution

First, graph the solution to the equation. Then use the **deSolve**(command to find an analytic solution.

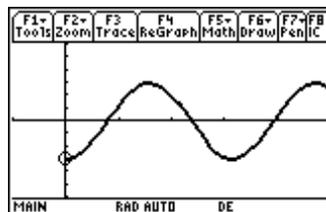
1. In the Y= Editor, use the substitutions described above and enter the equations and initial conditions shown at the right. Deselect y_2' by moving the cursor to y_2' line and pressing **F4** ✓.
2. In the Window Editor, enter the Window values as shown at the right.
3. Press **F1** **Tools**, and select **9:Format**, and make sure **Solution Method = RK** and **Fields = FLDOFF**.
4. Press **▣** **[GRAPH]** to graph the solution.



2. In the Window Editor, enter the Window values as shown at the right.
3. Press **F1** **Tools**, and select **9:Format**, and make sure **Solution Method = RK** and **Fields = FLDOFF**.



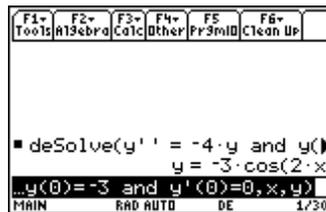
4. Press **▣** **[GRAPH]** to graph the solution.



5. Now enter the **deSolve**(command on the Home screen to find an analytic solution.

CATALOG **deSolve** (**Y** **2nd** **[']** **2nd** **[']** **▣** **4Y** **CATALOG** and **Y** **(** **0** **)** **▣** **(** **-** **)** **3** **CATALOG** and **Y** **2nd** **[']** **(** **0** **)** **▣** **0** **,** **X** **,** **Y** **)**

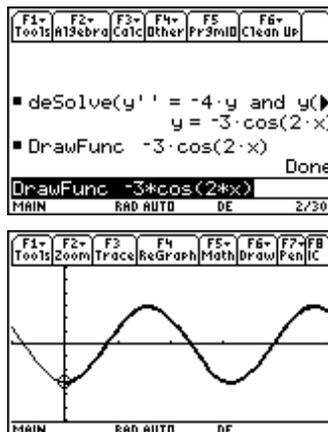
The analytic solution is $y = -3\cos(2x)$.



5. Use the **DrawFunc** command to compare the graph of the analytic solution with the graph obtained in differential equation graphing mode.

CATALOG **DrawFunc** **(-)** **3** **×** **2nd** **[COS]** **2X** **]** **ENTER**

6. Press **◀** **[GRAPH]**. The graph of $y = -3\cos(2x)$ enters the left part of the screen and continues over the graph produced by the differential equation grapher. This is evidence that you have the right solution.



Example 5: Modeling baseball trajectories

This example shows how to change the axes in differential equation graphing mode to model the path of a projectile.

A baseball is hit when it is 3 feet above the ground. It leaves the bat with an initial velocity of 152 ft/sec at an angle of 20° with the horizontal. How far will the ball travel?

Solution

The baseball moves both horizontally and vertically. Assume that the horizontal deceleration due to air resistance is directly proportional to the horizontal velocity. The vertical acceleration due to gravity is -32 ft/s^2 , and the vertical deceleration due to air resistance is proportional to the vertical velocity. Assume a constant of proportionality for air resistance of -0.05 . Make the following substitutions in order to enter this problem on the TI-89:

$$\text{horizontal distance} = x = y1$$

$$\text{initial horizontal distance} = yi1 = 0$$

$$\text{horizontal velocity} = y1' = y2$$

$$\text{initial horizontal velocity} = yi2 = 152 \cos(20^\circ)$$

$$\text{horizontal acceleration} = y1'' = y2' = -.05y2$$

$$\text{vertical distance} = y = y3$$

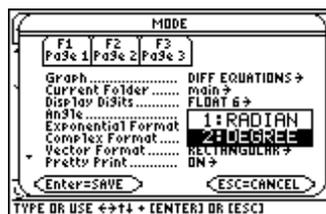
$$\text{initial vertical distance} = yi3 = 3$$

$$\text{vertical velocity} = y3' = y4$$

$$\text{initial vertical velocity} = yi4 = 152 \sin(20^\circ)$$

$$\text{vertical acceleration} = y4' = -32 -.05y4$$

1. In the MODE dialog box, set **Angle=DEGREE**.



2. In the Y= Editor, enter the equations and initial conditions.



3. Press **[F1] Tools** and select **9:Format**. Set **Solution Method = RK**. Press **[ENTER]**.

4. Specify the *x*- and *y*- axes by pressing **[2nd] [F7] Axes**. In the AXES dialog box, set the following:

Axes= CUSTOM
x Axis= y1
y Axis= y3

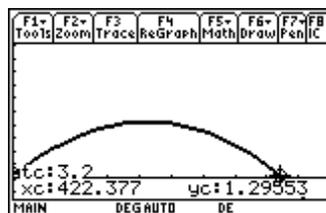


5. Press **[ENTER]** to save these settings. In the Window Editor, set the Window variables to the following values:

t0 =0	xscl =50
tmax =5	ymin = -15
tstep =.1	ymax =100
tplot =0	yscl =10
xmin =0	ncurves =0
xmax =500	difto1 =.001

6. Press **[GRAPH]** to graph the solution. The path should resemble the trajectory of a baseball. Press **[F3] Trace** to find the horizontal distance traveled by the ball.

The ball travels about 422 feet.



Exercises

1. Graph the slope field for the differential equation

$$\frac{dy}{dx} = \cos(x)$$

2. Graph the solution for the differential equation

$$\frac{dy}{dx} = \cos(x)$$

with initial condition $y(0)=0$. Graph the specific solution first with the slope field and then without it.

3. Graph the slope field for the differential equation

$$\frac{dy}{dx} = \frac{\sin(x)}{x}$$

4. Graph the solution for the differential equation

$$\frac{dy}{dx} = \frac{\sin(x)}{x}$$

with initial condition $y(-7)=-1$. Graph the specific solution first with the slope field and then without it.

Solve exercises 5 and 6 using tabular zoom.

5. How old is a sample in which 50% of the carbon-14 has decayed?
6. How old is a sample in which 80% of the carbon-14 has decayed?

Graph the solutions to the following higher-order differential equations (exercises 7 and 8) and where possible find the analytic solution. Compare the graph of the analytic solution with the graph made by the TI-89 in differential equation graphing mode.

7. $y'' = -9.8$, $y(0) = 0$, $y'(0) = 41$
8. $y'' = 1 - 2y - 2y'$, $y(0) = 0$, $y'(0) = 0$
9. Find the horizontal distance traveled by a baseball hit when it is 3 feet above the ground. The baseball leaves the bat with an initial velocity of 121 ft/sec at an angle of 36° with the horizontal. Assume the air resistance constant is -0.05 .
10. If there is a 10-foot fence 375 feet away in the outfield, is the ball in Exercise 9 a home run?