Math Objectives

- Students will discover the change of base rule for logarithms by examining the ration of two logarithmic functions with different bases.
- Students will review and use the properties of logarithms.
- Students will evaluate and simplify logarithmic expressions while using the change of base rule.
- Students will use appropriate tools strategically (CCSS Mathematical Practice).

Vocabulary

- logarithmic function
- change of base
- properties of logarithms

About the Lesson

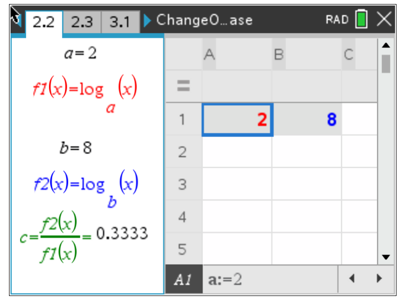
- This lesson involves manipulating logarithmic functions with different bases.
- As a result, students will:
- Use different properties of logarithms.
- Solve simple equations.
- Derive a formula that allows you to convert a logarithm to a different base..

TI-Nspire™ Navigator™

- Transfer a File.
- Use Class Capture to examine patterns that emerge.
- Use Live Presenter to demonstrate.
- Use Teacher Edition computer software to review student documents.
- Use Quick Poll to assess students' understanding

Activity Materials

Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

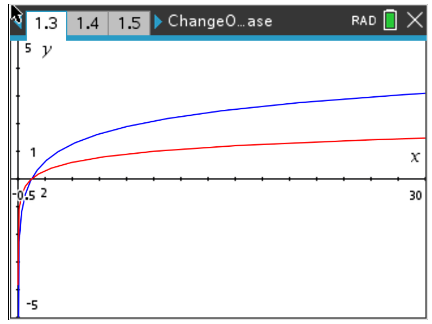


Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX II handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:
Student Activity
Change_of_Base_Nspire_Student.pdf
Change_of_Base_Nspire_Student.doc
ChangeOfBase.tns
ChangeOfBase_Sol.tns

In this activity, students discover the change of base rule for logarithms by examining the ratio of two logarithmic functions with different bases. It begins with a review of the definition of a logarithmic function, as students are challenged to guess the base of two basic logarithmic functions from their graphs. The goal of applying the properties of logarithms to add these two functions is introduced as a motivator for writing them in the same base. Students explore the hypothesis that the two functions are related by a constant first by viewing a table of values, then by exploring different values for the two bases. Finally, they prove the change of base rule algebraically and apply it to find the sum of the two original functions.



Teacher Tip: If you are using the *ChangeOfBase.tns* file with your students, Problem 1 follows along with pages 1.1 – 1.7, Problem 2 follows along with pages 2.1 – 2.3, and Problem 3 follows along with pages 3.1 – 3.7. Students can answer questions on the *tns* file or can just put the answers on this paper. The use of the graphs and tables on the *tns* file will be a nice visual for the students to use.

Problem 1 – Relating log functions with different bases

Open the file *ChangeOfBase.tns*. Move to page 1.3, you will see the graphs of two logarithmic functions with different bases:

$$\boldsymbol{f1(x) = \log_a(x)} \text{ and } \boldsymbol{f2(x) = \log_b(x)}.$$

(a) What are *a* and *b*? Trace the graphs to find out.

Solution: Students should trace the graphs (**menu**, **5 Trace**, **2 Trace All**) to find the values of a and b. $\boldsymbol{a = 3}$ and $\boldsymbol{b = 10}$.

(b) What points on the graph are the best clues to the base of the logarithmic function?

Solution: Students should realize that the most informative points on the graphs will be those at which y = 1 or some other whole number. When y = 1, x is equal to the base of the logarithm.

Suppose we are interested in the sum of these two functions,

$$\boldsymbol{(f1 + f2)(x) = \log_a(x) + \log_b(x).}$$

How could we write this as a single logarithmic expression?

- We can't apply the properties of logarithms unless the logarithms have the same base.
- We need to rewrite the functions with the same base.
- This means we want to find a function that is equal to $\boldsymbol{f1}$, but has a log base ***b*** instead of log base ***a***.

Maybe there is a constant *c* that could relate the two functions, like: $\boldsymbol{c \cdot f1(x) = f2(x)}$. Then, we would have $\boldsymbol{f1(x) = \frac{f2(x)}{c} = \frac{1}{c} \cdot \log_b(x)}$, which is a logarithmic function with base *b*.

We can't be sure there is such a constant, but that doesn't have to stop us from looking for one.

(c) Solve $\boldsymbol{c \cdot f1(x) = f2(x)}$ for ***c*** and enter the result in $\boldsymbol{f3(x)}$ on page 1.3 by pressing **tab**. What is ***c***? Press **ctrl T** to view a function table to see.

Solution: c = 0.47712

Problem 2 – A closer look at *c*

Is ***c*** always the same? Enter two new values of *a* and *b* in the gray cells on page 2.2. The value of *c* is calculated for you. Is it the same as in Problem 1? Record your results in the table below. Be sure to try some values of *a* and *b* such that one is a power of the other, like 2 and 8 or 3 and 9.

(d) Can you guess a formula for *c*?
Sample Table:

a	**f1(*x*)**	***b***	**f2(*x*)**	***c***
3	$\boldsymbol{\log_3 x}$	**27**	$\boldsymbol{\log_{27} x}$	**0.33333**
6	$\boldsymbol{\log_6 x}$	**36**	$\boldsymbol{\log_{36} x}$	**0.5**
2	$\boldsymbol{\log_2 x}$	**16**	$\boldsymbol{\log_{16} x}$	**0.25**
9	$\boldsymbol{\log_9 x}$	**3**	$\boldsymbol{\log_3 x}$	**2**

Formula: $\boldsymbol{c = \frac{1}{\log_a b}}$

Teacher Tip: In Problem 2, students will test different values of a and b (the bases of the logarithmic functions) to see how they affect the value of c. Students can work independently to enter values for a and b in the gray cells in the first row of page 2.2, then record a, b, and the value of c that is calculated for them in the lower rows. This creates a function table that gives students an intuitive sense of the multivariable function c(a, b). Guide students to guess a rule for c based on this data. If you wish, direct them to calculate 1/c in Column D as a way of giving a hint. $\left(c = \frac{1}{\log_a b}\right)$.

Problem 3 – Deriving the Change of Base Rule algebraically

We are convinced now that there is a constant that relates $\boldsymbol{\log_a(x)}$ to $\boldsymbol{\log_b(x)}$ and that the constant depends on the values of ***a*** and ***b***. We may even have an idea what the constant is. Time to use some algebra to find out for sure.

Two functions are equal if and only if their values are equal for every *x*-value in their domain. Let's pick a point (*x*, *y*) on the graph of $\boldsymbol{f1(x)}$. For this (*x*, *y*), $\boldsymbol{\log_a(x) = y}$. If we can write ***y*** in terms of logs base ***b***, we will have our function.

(e) Rewrite $\log_a(x) = y$ as an exponential function.

Solution: $a^y = x$

(f) We want an expression with base ***b*** log, so take $\log_b$ of both sides.

Solution: $\log_b a^y = \log_b x$

(g) Simplify using the properties of logs. Solve for ***y***.

Solution: $y \cdot \log_b a = \log_b x$

$$y = \frac{\log_b x}{\log_b a}$$

(h) What is ***c***?

Solution: $c = \dfrac{1}{\log_a b}$

Check your equation for ***c*** against the value of *c* that you collected earlier. Enter it in the formula bar of Column D on page 2.2. When prompted, choose **Column Reference**, not Variable, for *a* and *b*.

(i) Is the equation correct? Explain.

Possible Solution: Yes, this is correct. After comparing the different values of a and b used earlier, each time has resulted in a c value based on the formula above.

You have found a formula for changing the base of a logarithm. To change a log base ***a*** expression to log base ***b***, simply divide the expression by $\log_a(b)$. This can be written as

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}.$$

(j) Use this formula to find $(f1 + f2)(x)$ if $f1(x) = \log_3(x)$ and $f2(x) = \log_{10}(x)$.

Solution:

$$\begin{aligned}(f1 + f2)(x) &= \log_3 x + \log_{10} x \\ &= \frac{\log_{10} x}{\log_{10} 3} + \log_{10} x \\ &= \frac{1}{\log_{10} 3}(\log_{10} x + \log_{10} 3 \cdot \log_{10} x) \\ &= \frac{1}{\log_{10} 3}\left(\log_{10} x + \log_{10} x^{\log_{10} 3}\right) \\ &= \frac{\log_{10} x^{1 + \log_{10} 3}}{\log_{10} 3}\end{aligned}$$

Teacher Tip: Problem 3 steps students through the process of deriving the Change of Base rule algebraically. You may wish to have students record their work on a piece of paper instead of typing their expressions into the handheld.

Problem 4 – Further practice with the Change of Base Rule

(k) Use the Change of Base Rule to simplify the expression: $\frac{1}{4}\log_9 27$.

Solution: $\frac{1}{4}\log_9 27 = \frac{1}{4} \cdot \frac{\log_3 27}{\log_3 9} = \frac{1}{4} \cdot \frac{3}{2} = \frac{3}{8}$

(l) The function f is given by $f(x) = \log_4(x)$. Without a handheld, use the Change of Base Rule to evaluate $f(8)$.

Solution: $f(8) = \log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2}$

(m) If $f(x) = \log_8 x$ and $g(x) = \log_{64} x$, given that the input values into each function are equal, describe the relationship between the output values of $f(x)$ and $g(x)$.

Solution: Using the Change of Base Rule, the outputs of g(x) will be one half the outputs of f(x).

$$\log_{64} x = \frac{\log_8 x}{\log_8 64} = \frac{f(x)}{2}$$